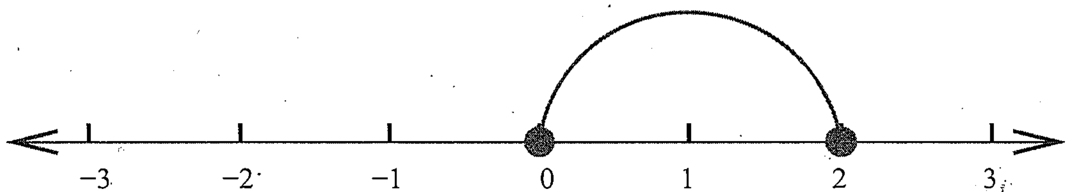
## 2.5 Intervals on the real line

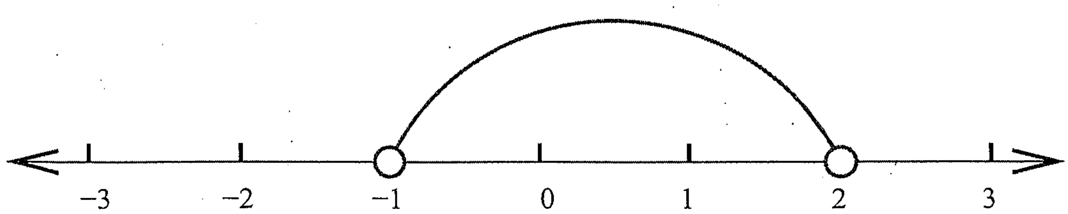
- On Page 12 we briefly encountered number lines (or *real* lines) and order (such as 'less than', written $<$).
- Any real number can be marked as a **single point** on the real line.
- **Intervals** or **regions** can also be marked on the real line. An interval includes *all real numbers which lie between two endpoints*.
- Such intervals can be described by *inequalities*, using the signs: $<$ $\leq$ $>$ $\geq$

***Example** 2.5.1* On the real line, mark the interval corresponding to $x \geq 0$ and $x \leq 2$.




We have highlighted the region between $x = 0$ and $x = 2$, with a solid black circle at each end point, and a (curved) line between the end points. This is used to denote **every point** between 0 and 2 (inclusive).

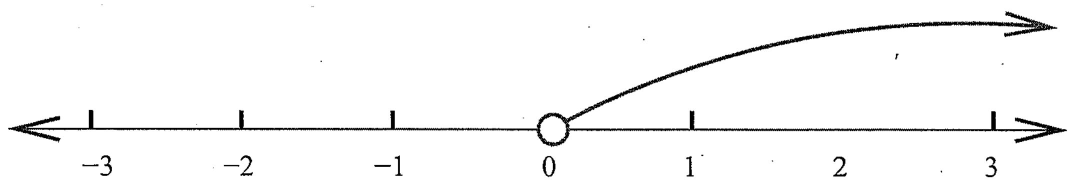
***Example** 2.5.2* On the real line, mark the interval corresponding to $x > -1$ and $x < 2$.




Now we have used a non-filled circle at each end point. This is used to denote **every point** between $-1$ and 2, but **not** including $-1$ and 2.

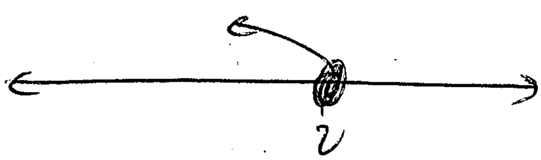
- Make sure you understand the difference between $\leq$ and $<$, and between $\geq$ and $>$
  - For $\leq$ and $\geq$ the endpoint occurs **inside** the interval, and is marked with a solid circle.
  - For $<$ and $>$ the endpoint occurs **outside** the interval, and is marked with a non-filled circle.
- Some intervals only have one endpoint (e.g. $x > 4$).
- This means that the interval goes on forever in one direction. If it goes to the right then we say it goes to infinity, written $\infty$. If it goes to the left, we say it goes to negative infinity, written $-\infty$.
- This is marked on a real line by an arrow pointing in the correct direction.

***Example** 2.5.3* On a real line, mark the region $x > 0$.

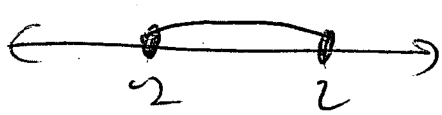



***Question** 2.5.4* Mark each of the following intervals on the real line:

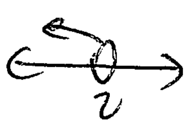
(1) $x \leq 2$



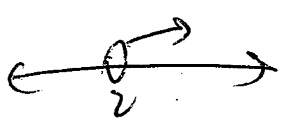

(2) $-2 \leq x \leq 2$ (This means $-2 \leq x$ and $x \leq 2$.)




(3) $x < 2$



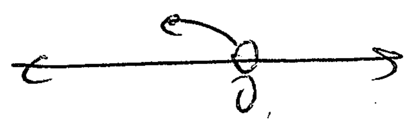

(4) $x > 2$



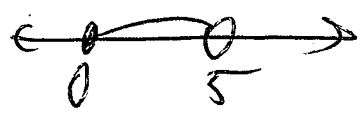
- There is an easy way to write intervals:
  - $[a, b]$ denotes the interval $a \leq x \leq b$
  - $[a, b)$ denotes the interval $a \leq x < b$
  - $(a, b]$ denotes the interval $a < x \leq b$
  - $(a, b)$ denotes the interval $a < x < b$
- $a$ and $b$ are called the **endpoints** of the interval. Note that $a$ (the first endpoint) is **always less than or equal to** $b$.
- Note the brackets: they indicate the type of interval.
  - A square bracket means the corresponding endpoint falls **inside** the interval. On the real line, the endpoint is marked with a solid circle.
  - A round bracket means the corresponding endpoint falls **outside** the interval. On the real line, the endpoint is marked with a non-filled circle. **(Note that $-\infty$ and $\infty$ always have a round bracket, not a square bracket.)**
- Be clear on what happens when an endpoint is **outside** an interval, e.g. $x > 0$. The point $x = 0$ is not in the interval, but every value greater than 0 is in the interval. So 0.5, 0.01, 0.000001 and 0.00000001 are all in the interval.

***Question** 2.5.5* Write each of the following intervals using inequality signs, and then mark each one on a real line:

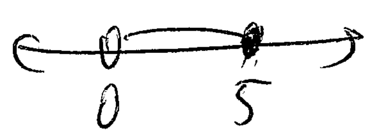
(1) $(-\infty, 0)$ $x < 0$




(2) $[0, 5)$ $0 \leq x < 5$




(3) $(0, 5]$ $0 < x \leq 5$



## 2.6 Solving inequalities

- We know how to solve equations with an "=" sign.
- The key rule was: whatever you do to one side, you must also do to the other side.
- We can also solve *inequalities*, which look like equations but instead have signs like $<$ or $\geq$.
- There are two major differences between equations and inequalities:
  - the answer to most inequalities is an **interval**, not a single point; and
  - the rules for manipulating inequalities are a bit different to those for solving equations.

**Rules for solving inequalities.**

1. *The same quantity can be added to, or subtracted from, both sides of the inequality.*
2. *Both sides of the inequality can be multiplied by, or divided by, the same* **positive** *quantity.*
3. *If both sides are* **multiplied** *by, or* **divided** *by, the same* **negative** *quantity, then the inequality must be reversed (that is, $<$ becomes $>$, $>$ becomes $<$, and so on).*
4. *If $a < b$ then $b > a$; if $a > b$ then $b < a$.*
   *If $a \leq b$ then $b \geq a$; if $a \geq b$ then $b \leq a$.*

- Rules 1 and 2 are the same as for solving equations.
- Pay particular attention to Rules 3 and 4: the inequality sign must be **reversed** when applying these rules!

## 2.7 Square roots

- We have previously seen square roots, written with a $\sqrt{\phantom{a}}$ sign. If $a$ is a real number then we know that:
  1. $\sqrt{a}$ is only defined if $a \geq 0$
  2. $\sqrt{a} \times \sqrt{a} = a$
  3. if $a > 0$ then $a$ has two square roots, one positive and one negative.
- To avoid confusion, $\sqrt{a}$ is usually taken to mean the positive square root of $a$.
- To get the negative square root, write $-\sqrt{a}$.

The following rules allow us to simplify square roots.

**Important properties of square roots.**

*If $a$ and $b$ are real numbers with $a \geq 0$ and $b \geq 0$, then*

*(1)* $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b} = \sqrt{ab}$

*(2)* $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

***Example** 2.7.1*

1. $\sqrt{4} \times \sqrt{4} = \sqrt{4 \times 4} = \sqrt{16} = 4$ and $\sqrt{7} \times \sqrt{7} = 7$
2. $\sqrt{5} \times \sqrt{20} = \sqrt{5 \times 20} = \sqrt{100} = 10$
3. $\sqrt{\dfrac{4}{9}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}$
4. $-\sqrt{16} = -4$

***Example 2.6.1*** Solve the inequality $-3x+2 \leq 6-x$.

$$-3x+2 \leq 6-x$$

so $$-3x+2+x \leq 6-x+x$$

so $$-2x+2-2 \leq 6-2$$

so $$\boxed{-2x \leq 4}$$

so $-2x \div -2 \geq 4 \div -2$ (the inequality is reversed)

so $$x \geq -2$$

$2 \leq 6$

$-2$

$[-2, \infty)$.

***Question 2.6.2*** Find all $x$ which satisfy $2x-4 > x+3$. Write your answer in interval format and mark it on the real line.

$2x-4 > x+3$

$x-4 > 3$

$x > 7$

$(7, \infty)$

$7$

✓

***Question 2.6.3*** Find all $x$ which satisfy $-2x \leq x+3$. Write your answer in interval format and mark it on the real line.

$-2x \leq x+3$ or $-3x \leq 3$

$0 \leq 3x+3$ $x \geq -1$

$-3 \leq 3x$

$x \geq -1$

$[-1, \infty)$

$-1$

✓

***Question 2.6.4*** Find all $y$ which satisfy $3(y+2) < 3y+4$.

$\cancel{3y}+6 < \cancel{3y}+4$

$\cancel{-3y}$ $\cancel{-3y}$

$6 < 4$. $\therefore$ No values for $y$.

**Question 2.7.2** Simplify $\dfrac{\sqrt{8}\times\sqrt{6}}{\sqrt{16}}$. $= \dfrac{\sqrt{8}\sqrt{6}}{4} = \dfrac{\sqrt{48}}{4}$

$= \dfrac{\sqrt{16}\sqrt{3}}{4} = \dfrac{4\sqrt{3}}{4} = \sqrt{3}$

- There are some common errors with square roots.
- Pay attention to the following facts; they each say that the two quantities are **not equal**.

**Non-properties of square roots.**

(1) $\sqrt{a}+\sqrt{b} \neq \sqrt{a+b}$ $\quad \sqrt{9}+\sqrt{2} \neq \sqrt{11}$

(2) $\sqrt{a}-\sqrt{b} \neq \sqrt{a-b}$

***Example*** 2.7.3 Make sure you understand that:

$$\sqrt{2x}\times\sqrt{3y} = \sqrt{6xy}$$

but you cannot simplify:

$$\sqrt{2x}+\sqrt{3y}$$

***Question*** 2.7.4 By letting $a = 9$ and $b = 16$, show that it is **not true** that $\sqrt{a}+\sqrt{b} = \sqrt{a+b}$.

$LHS = \sqrt{9}+\sqrt{16}$ $\quad RHS = \sqrt{9+16}$

$= 3+4$ $\quad = \sqrt{25}$

$= 7$ $\quad = 5$

$LHS \neq RHS$

## Surds

- Some square roots can be written exactly as fractions; that is, they are **rational numbers**.

> *Example 2.7.5* The following square roots are rational:
>
> $$\sqrt{4} = 2 = \frac{2}{1} \qquad \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$$

- Many square roots **cannot** be written exactly as fractions; that is, they are **irrational numbers**.
- For example, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$ are all irrational, and there is no way of writing them more simply.
- Irrational square roots are called **surds**. 1, $\frac{1}{2}$ $\sqrt{4}=2$
- Sometimes, a surd can be written in a simpler form, by using the properties of square roots. In particular:

(1) $\sqrt{a^2} = a$ (for example, $\sqrt{16} = \sqrt{4^2} = 4$) and

(2) $\sqrt{a \times a} = a$ (for example, $\sqrt{2 \times 2} = 2$).

- These rules let us 'take things outside' the square root.

### Simplifying square roots

*Given a square root, we usually write it in simplest form by trying to 'take something outside' the square root sign. This is done via the following process:*

- *Factor the number inside the square root sign, looking for*
  - *factors that are square numbers (e.g. $4, 9, 16, \ldots$); or*
  - *pairs of identical factors (if you don't easily find a square factor).*
- *Use rules (1) and (2) above to simplify.*

***Example 2.7.6*** Write $\sqrt{12}$ in simplest form. $\sqrt{9} = 3$ $\sqrt{16} = 4$.

Notice that 4 is a square number and that $12 = 4\times 3$. So:

$$\sqrt{12} = \sqrt{4\times 3} = \sqrt{4}\times\sqrt{3} = 2\times\sqrt{3} = 2\sqrt{3}$$

Alternatively,

$\sqrt{4}$

$$\sqrt{12} = \sqrt{2\times 6} = \sqrt{2\times 2\times 3} = \boxed{\sqrt{2\times 2}}\times\sqrt{3} = 2\times\sqrt{3} = 2\sqrt{3}$$

***Question 2.7.7*** Simplify $\sqrt{20}$.

$$\sqrt{20} = \sqrt{4\times 5} = \sqrt{4}\times\sqrt{5} = 2\sqrt{5}$$

$$= \sqrt{10\times 2} = \sqrt{5\times 2\times 2} = \sqrt{5}\times\sqrt{4} = 2\sqrt{5}.$$

## Arithmetic on surds

- Surds can be involved in expressions. For example, $3+\sqrt{5}$ is an expression involving a surd.
- Mathematical operations (such as addition, multiplication and so on) can be performed on such expressions.
- Be careful to remember BEDMAS and the relevant properties of square roots.

***Example 2.7.8***

$$\begin{aligned}(\sqrt{2}+5)+(\sqrt{2}-6)-\sqrt{2} &= \sqrt{2}+\sqrt{2}-\sqrt{2}+5-6\\ &= \sqrt{2}-1\end{aligned}$$

$\sqrt{3} + 4\sqrt{3} - 6\sqrt{3} = -\sqrt{3}.$



$x + 4x - 6x$ where $x = \sqrt{3}$

**Example 2.7.9**

$$\begin{aligned}
& 3\sqrt{2}\times 5\sqrt{6}+10\sqrt{3} \\
= \; & 15\times\sqrt{2}\times\sqrt{6}+10\sqrt{3} \\
= \; & 15\sqrt{12}+10\sqrt{3} \\
= \; & 15\times 2\sqrt{3}+10\sqrt{3} \\
= \; & 30\sqrt{3}+10\sqrt{3} \\
= \; & 40\sqrt{3}
\end{aligned}$$

**Question 2.7.10** Show that $\sqrt{2}+\sqrt{2}+\sqrt{2}=\sqrt{18}$.

$LHS = \sqrt{2}+\sqrt{2}+\sqrt{2}$

$= 3\sqrt{2}$

$RHS = \sqrt{18}$

$= \sqrt{9}\sqrt{2}$

$= 3\sqrt{2}$

$= LHS$

**Question 2.7.11** Simplify $(\sqrt{8}-\sqrt{2})(\sqrt{2}+\sqrt{6})$.

$= \sqrt{16}+\sqrt{48}-\sqrt{4}-\sqrt{12}$

$= 4+\sqrt{16}\sqrt{3}-2-\sqrt{4}\sqrt{3}$

$= 2+4\sqrt{3}+2\sqrt{3}$

$= 2+6\sqrt{3}$

## 2.8 Powers and Exponents

- On Page 25 we briefly encountered *exponentiation*.
- For example, $3^2 = 3 \times 3$.
- In the expression $3^2$, 3 is called the *base* and 2 is called the *power*, *exponent*, or *index*.
- There are various rules that allow us to simplify operations involving powers. You must be familiar with these rules.

**Power Rule 1: Product of powers**

*If $a$, $m$ and $n$ are real numbers, then:*

$$a^m \times a^n = a^{m+n}$$

*Note that in this rule, the base must be the same in* **both** *places on the LHS of the equals sign* **and** *on the RHS.*

***Example*** *2.8.1*

- $2^2 \times 2^3 = 2^{2+3} = 2^5 = 32$
  You can see why the rule works:
  $2^2 \times 2^3 = (2 \times 2) \times (2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$
- $y^3 \times y^2 \times y = y^{3+2+1} = y^6$
- We cannot simplify $x^3 \times y^2$ as the first base $x$ is not the same as the second base $y$. The most we can do is simplify it to $x^3y^2$.

***Question** 2.8.2* Simplify each of the following:

**(a)** $3^4 \times 3^2 \times 3^3$ $= 3^9$

**(b)** $x^7 \times x^2 \times y^4 \times x^6$ $= x^{15} y^4$

**(c)** $2^n \times 2^3$ $= 2^{n+3}$ $\neq 2^{3n}$

## Power Rule 2: Dividing powers

*If $a, m, n$ are real numbers, with $a$ non-zero, then:*

$$a^m \div a^n = a^{m-n}$$

*Just as in Rule 1, the base must be the same in* **both** *places on the LHS of the equals sign* **and** *on the RHS.*

***Example** 2.8.3*

- $3^5 \div 3^2 = 3^{5-2} = 3^3 = 27$

  You can see why the rule works:

  $$3^5 \div 3^2 = \frac{3\times3\times3\times\cancel{3}\times\cancel{3}}{\cancel{3}\times\cancel{3}} = 3\times3\times3 = 3^3$$

- $p^{10} \div p^6 = p^{10-6} = p^4$

***Question** 2.8.4* Simplify each of the following:

**(a)** $(-7)^{12} \div (-7)^5 \div (-7)^3$ $= -7^7 \div -7^3 = -7^4.$

**(b)** $x^4 \div x^{-4}$ $= x^8$ $\left(= x^{4--4}\right).$

**(c)** $3^{n+4} \div 3^{n+2}$ $= 3^{\cancel{n}+4-(\cancel{n}+2)}.$ $= 3^2 = 9$

$-(n+2)$

$= -n-2$

## Power Rule 3: Power equal to 0 or 1

*If $a$ is any non-zero real number then:*

$$a^0 = 1 \qquad \text{and} \qquad a^1 = a$$

**Example** 2.8.5

- $3^2 \div 3^2 = 1$ and $3^2 \div 3^2 = 3^{2-2} = 3^0$. Demonstrating $3^0 = 1$.
- $x^4 \div x^3 = \dfrac{x \times x \times x \times x}{x \times x \times x} = x$ and $x^4 \div x^3 = x^{4-3} = x^1$.

  So it must be that $x^1 = x$.

**Question** 2.8.6 Simplify each of the following:

**(a)** $(2^{52} \times (-0.14536)^5)^0 \quad = 1.$

**(b)** $x^2 \times x^1 \times x^3 \div x^5 \quad = x^6 \div x^5 = x^1 = x.$

**(c)** $x^2 \times x^0 + x^3 \times y^0$

$= x^2 + x^3 \times 1$

$= x^2 + x^3 \neq x^5$

## Power Rule 4: Negative power

*Let $a$ be any non-zero real number and $m$ be any real number, then:*

$$\boxed{a^{-m} = \frac{1}{a^m}}$$

*Note that the expression $a^{-m}$ has been rewritten as a fraction and the power is now 'positive' $m$.*

*Example 2.8.7*

- $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$.

  You can see why the rule works:

  $$10^3 \div 10^5 = \frac{\cancel{10}\times\cancel{10}\times\cancel{10}}{\cancel{10}\times\cancel{10}\times\cancel{10}\times 10\times 10} = \frac{1}{10\times 10} = \frac{1}{10^2}.$$

  But $10^3 \div 10^5 = 10^{3-5} = 10^{-2}$.

  Hence $10^{-2} = \dfrac{1}{10^2}$.

- $x^{-3} = \dfrac{1}{x^3}$.

- $\dfrac{1}{5^{-2}} = \dfrac{5^2}{1} = 25$.

*Question 2.8.8* Simplify each of the following:

(a) $2^{-1}\times 10 \quad = \dfrac{1}{2^1} \times 10 = \dfrac{1}{2}\times 10 = 5$

(b) $7^{-2}\times 14 \quad = \dfrac{1}{7^2}\times 14 = \dfrac{1}{49}\times\dfrac{14}{1} = \dfrac{14}{49} = \dfrac{2}{7}$

(c) $\dfrac{x^5}{1}\times\dfrac{1}{x^4} \quad = \dfrac{x^5}{x^4} = x^1 = x \quad \| \quad x^5 \times x^{-4} = x^1 = x.$

**Power Rule 5: Fractional powers**

*Let $a$ be a real number and $m$ be a non-zero real number, then:*

$$a^{1/m} = \sqrt[m]{a}$$

*In particular, for $m = 2$ we have $a^{1/2} = \sqrt[2]{a} = \sqrt{a}$. (For some values of $m$ there are restrictions on allowed values of $a$. For example, if $m = 2$ then $a$ cannot be negative.)* $9^{1/2} = \sqrt{9} = 3$.

*Example* 2.8.9

- $9^{1/2} = \sqrt{9} = 3.$

  You can see why the rule works:

  $9^{1/2} \times 9^{1/2} = 9^{1/2+1/2} = 9^1 = 9.$

  Hence $9^{1/2}$ must be $\sqrt{9}$.

- $(x^{1/2})^2 = (\sqrt{x})^2 = x.$
- $7^{1/3} \times 7^{1/3} \times 7^{1/3} = 7^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 7^1 = 7.$

$\sqrt{a} = a^{1/2}$

$\sqrt{x} \times \sqrt{x} = x. \quad (= \sqrt{x^2})$

***Question*** 2.8.10 Simplify each of the following:

**(a)** $(3^2 \times 4^{1/2})^{1/2}$ $= (9 \times \sqrt{4})^{1/2} = \sqrt{9 \times 2} = \sqrt{18}$
$= \sqrt{9} \times 2$
$= 3\sqrt{2}$

**(b)** $x^{-1/2} - \dfrac{\sqrt{x}}{x}$

$= \dfrac{1}{x^{1/2}} - \dfrac{x^{1/2}}{x^1} = \dfrac{1}{\sqrt{x}} - x^{\frac{1}{2}-1} = \dfrac{1}{\sqrt{x}} - x^{-1/2} = \dfrac{1}{\sqrt{x}} - \dfrac{1}{x^{1/2}}$
$= \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{x}}$
$= 0$

**Power Rule 6: Powers raised to powers**

*If $a$, $b$, ~~$m$ and $n$~~ are real numbers ($b \neq 0$ in the fraction) then:*

$$(a^m)^n = a^{mn} \qquad (ab)^n = a^n b^n \quad \text{and} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

*Example* 2.8.11

- $(4^2)^3 = 4^{2\times 3} = 4^6$

  You can see why the rule works:

  $(4^2)^3 = 4^2 \times 4^2 \times 4^2 = 4^{2+2+2} = 4^{2\times 3} = 4^6$

- $(x^2y)^2 = x^{2\times 2}y^{1\times 2} = x^4y^2$