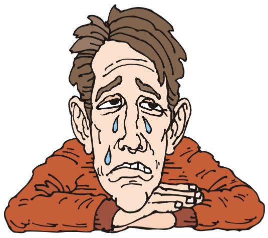
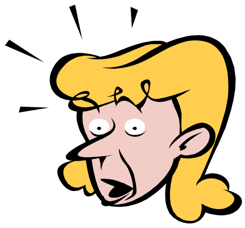

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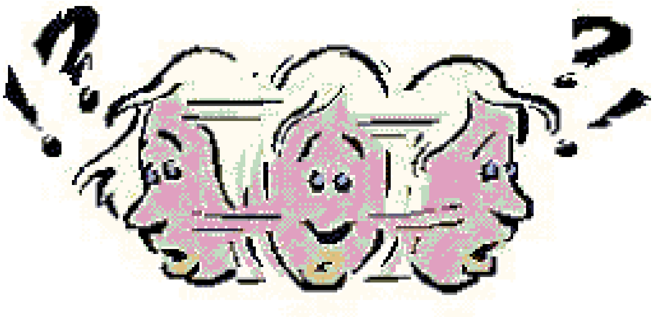
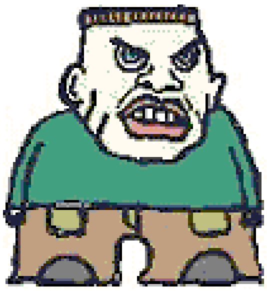
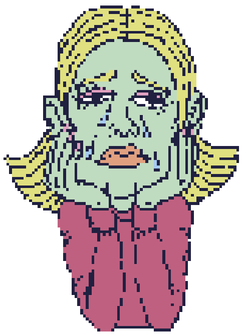
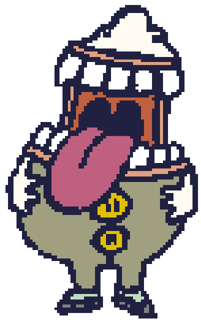
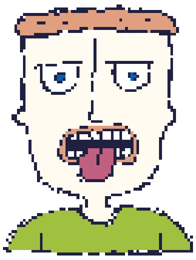


Lecture notes


Tenth edition, 2011

About these notes

These are the lecture notes for MATH1040 (and for MATH7040, for students enrolled in the Graduate Certificate/Master of Education. Throughout these notes, we will only refer to MATH1040; however, any comments are also relevant for MATH7040).

We will use these notes very heavily, so it is important that you get your own copy. Details on how you can obtain a copy are in the course profile and will also be given in class during the first week of semester. Please note that there is no text book for MATH1040, so these notes are your primary source of information. Don't re-use a copy from previous years or from your friends: the notes change from year to year, and in addition it is important for you to write things in your own words.

In lectures, we will use visualisers and notes. These lecture notes contain copies of all the pages covered in lectures. Thus you will have time to listen and think in class, rather than spending your whole time writing. However, there are many spaces in your notes for questions and solutions. We'll work through these in lectures, and you should write down all the information given.

Each year, some people accidentally lose their notes, which causes big problems for them. You might like to write your name and some contact details on the bottom of this page just in case.

Other materials for MATH1040 include a study guide (which contains extra information, including previous exam papers and solutions), and assignments.

These notes have been prepared very carefully, but there will inevitably be some errors in them. We are continually trying to improve the notes. If you have any suggestions on how to do so, please tell us.

These important notes belong to: ______________________________

If you find them, please return them to me!

I can be contacted via: ______________________________

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How to use these notes

These notes are organised into the following main components:

- *general notes*, which outline background material and introduce new ideas;
- *key points*, which summarise key definitions and concepts;
- *examples*, which give fully-worked examples showing how to solve important problems;
- *questions*, which you can try to solve yourself, and will be completed in class;

These components all look a bit different, to make them easier to find. A brief example of each one is:

- General notes (involving words and mathematical content) are often written with bullet points.

Key points

Key points are written in boxes with rounded corners, like this, with the title identifing the key point.

Example *0.0.1* How do worked examples look?

Answer: Worked examples are written in "double" boxes, like this, with the question followed by the solution.

Question *0.0.2* Questions are written like this, in bold boxes. In each case there is room to write in the answers (including working). We'll complete these questions in lectures, which will often involve you first doing some working yourself and then some class discussion.

Table of contents.

10.2 More trigonometry 193

1 Numbers and arithmetic

Why are we covering this material?

- This material is fundamental to all the maths you will do in MATH1040, and in many courses you'll take at University.
- It's even useful in everyday life!
- Many of you will be quite familiar with this material, but many others will not.
- If you can't do this stuff, you won't be able to do the harder and more interesting things.
- Try to stay awake! Things get much harder fairly quickly.
- Later in semester, many people will have problems with these introductory concepts.
- **Topics in this section are**
  - Thinking about Maths.
  - Types of numbers.
  - Number lines and order.
  - Absolute values.
  - Simple mathematical operations.
  - Order of operations.
  - Prime numbers and factors.
  - Fractions.
  - Introduction to exponentiation.
  - Square roots.

1.1 Thinking about Maths

- Mathematics is often not easy, but it's important!
- You'll encounter some sort of maths almost every day of your life, at the shops, at a football ground, as you travel, and as you earn (and spend!) money.
- Most people need to think quite hard when doing maths, but there are some skills and tricks that really help.
- Two important approaches you must learn to use are **estimating** and **checking your answers**.

Estimation and approximation

Often, it's useful to quickly estimate a rough answer. You can do this by approximating some of the numbers, thus simplifying the calculations. Your answer will not be exactly correct, but it should be "close" to the real answer.

The context of the question will determine how accurate you need to be. Sometimes it's good enough to be quite rough.

***Example** 1.1.1* Peter works 36.25 hours per week, and earns \$32.6174 per hour. Estimate his weekly income.

Answer: Working approximately 40 hours per week, earning approximately \$30 per hour, equals \$1200.

For reference, the exact answer is \$1182.38. Note that we rounded the number of hours up, and the hourly pay-rate was rounded down; this helped to increase the accuracy of the estimate.

Question *1.1.2* One Australian dollar (AUD) is worth $0.878 United States dollars (USD). Big Bad John spends 10 nights at a hotel in Las Vegas, at 93 USD per night. His credit card company charges a 1.5% fee to convert from USD to AUD. Roughly estimate his bill in AUD.

Question *1.1.3* Roughly estimate the number of babies born in Australia each year.

- Most maths questions have one correct final answer, but many different ways of getting that answer.
- Usually, you can use any valid method you like to get the answer, although sometimes you'll be asked to use a particular method.
- It's important that you show the steps you take, as many marks will usually be allocated to your working.

Checking your answer

It's easy to make mistakes when answering a question. Whenever possible, you should check your answer. Ways of doing this include:

- *Where appropriate, ask yourself: 'Does the answer make sense in a real-life context'? (But be careful doing this!)*
- *Use estimation to check whether the answer is 'plausible'.*
- *Check each of the steps in your working.*

Also, don't readily trust your calculator - it does exactly what you tell it to do, so make sure you press the right keys!

Question *1.1.4* For each question, decide which answer(s) are most likely to be correct. Explain why.

1. In an Olympic 400m running race, the maximum speed attained by the winner (in metres per second) is:

(a) 63.7 **(b)** 10 **(c)** 2

2. \$1000 is invested in a bank account earning 8% interest per annum for 3 years. What is the final balance:

(a) \$16728.33
(b) \$827.67
(c) \$1259.71
(d) \$1412.68

Be careful how you use estimation. In most cases, problems need **exact** answers. Estimation can be used to check, but shouldn't be used to find the actual answer.

This section might seem easy, but it's so important that we covered it first. For the rest of this course (and others), make sure you use estimation, and always check your answers.

1.2 Types of numbers

- $\mathbb{N}$ natural numbers
  - "counting" numbers
  - $1, 2, 3, 4, \ldots$
  - examples of uses:
    * counting pigs in a (non-empty) pen
    * counting how many birthdays you've had

- $\mathbb{Z}$ integers
  - positive and negative numbers without decimals
  - $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
  - examples of uses:
    * measuring a credit-card balance in cents
    * counting number of seconds before or after a rocket takes off.

- $\mathbb{Q}$ rational numbers
  - quotient of integers
  - those numbers which can be written exactly as a fraction
  - For example,

$$\begin{array}{cccc} -3, & \frac{1}{4} = 0.25, & -\frac{7}{5} = -1.4, & 0 = \frac{0}{1} \\ 4, & 1000, & \frac{49}{50}, & -4\frac{1}{2} \end{array}$$

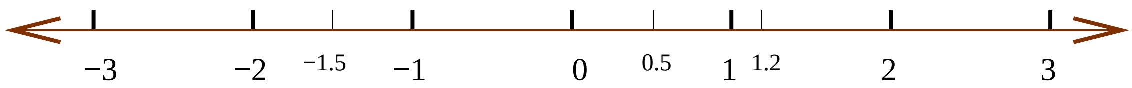
  - examples of uses:
    * measuring blood alcohol content
    * cutting a birthday cake into pieces, with sizes proportional to ages

- $\mathbb{R}$ real numbers
  - includes rationals and irrationals
  - Irrationals: numbers which are *not* rational (and hence cannot be written exactly as a fraction).
  - e.g. $\pi = 3.14159\ldots$

    $\sqrt{2} = 1.4142\ldots$

    also e, $\sqrt{5}$, π^2
  - examples of uses:
    * finding the area of an oval
    * calculating length of the fence of a round yard.

1.3 Number lines and order



- A number line (sometimes called a *real line*) shows the order of real numbers; given any two real numbers, the one to the right is 'greater than' the one to the left.
- Alternatively, the left one is 'less than' the one to the right.
- Every real number occurs somewhere on the number line; we often just mark integers.
- Loosely, we think of the number line going from $-\infty$ (negative infinity) on the left to ∞ (infinity) on the right.

Given two numbers, there are 5 common ways of writing the relationship between the numbers:

$=$	$<$	$>$	$\leq$	$\geq$
equal to	less than	greater than	less than or equal to	greater than or equal to

Example *1.3.1*

$$-3 < 1 \qquad 1 > -3 \qquad -3 \leq 1 \qquad 1 \geq -3$$

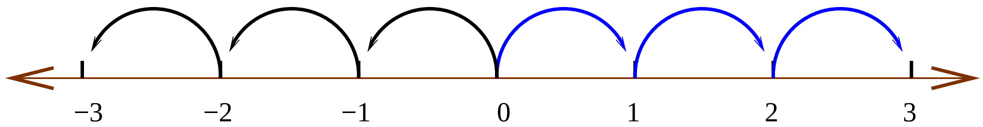
$$-2 \leq -2 \qquad -2 \geq -2 \qquad -2 = -2$$

$$\frac{1}{4} < \frac{1}{2} \qquad \frac{1}{2} \geq \frac{1}{4} \qquad \frac{1}{4} = 0.25$$

1.4 Absolute values

- On a number line, all numbers to the left of zero are negative, written with a $-$ sign before them (eg -3).
- There is a special relationship between a number and its negative: both are exactly the same distance from 0 (in opposite directions).

Example *1.4.1* -3 and 3 are both a distance of 3 from the point 0. 3 is three steps to the right of 0, -3 is three steps to the left of 0.



$-(-3) = 3$ (Note: $-0 = 0$)

Given a number, sometimes we are interested in **how far the number is from 0**, but we don't care in what direction.

Absolute value.

The absolute value of a number is its distance from zero.
If x is any number then we write $|x|$ to represent the absolute value of x.
Note that absolute value is always positive or 0.

Example *1.4.2* Understand the following examples.

$$|2| = 2 \quad |-2| = 2 \quad |-3\tfrac{1}{2}| = 3\tfrac{1}{2} \quad |0| = 0 \quad -|5| = -5$$

Question *1.4.3* Evaluate each of the following:

(1) $|-7.82|$

(2) $-|-1|$

(3) $|-2+5|$

1.5 Simple mathematical operations

- You should be quite familiar with the following operations:

$+$	addition	$-$	subtraction
$\times$	multiplication	$/$ or $\div$	division
$(\ )$ or $[\ ]$	brackets		

- Note that subtraction is the same as adding the negative; for example, $3 - 4 = 3 + (-4)$.

- When multiplying or dividing negative numbers, be careful! The rules are:

1st number	$\times$ or $\div$	2nd number	answer
$+$ve	$\times$ or $\div$	$+$ve	$+$ve
$+$ve	$\times$ or $\div$	$-$ve	$-$ve
$-$ve	$\times$ or $\div$	$+$ve	$-$ve
$-$ve	$\times$ or $\div$	$-$ve	$+$ve

Example *1.5.1*

$$\begin{array}{rclrcl} 4 \times 3 &=& 12 & 6 \div 3 &=& 2 \\ 4 \times -3 &=& -12 & 6 \div -3 &=& -2 \\ -4 \times 3 &=& -12 & -6 \div 3 &=& -2 \\ -4 \times -3 &=& 12 & -6 \div -3 &=& 2 \end{array}$$

- Be careful when dealing with zero:
  - It is never possible to divide by zero.
  - Zero divided by any non-zero number equals zero.
  - Any number $\times$ zero equals zero.
  - If two numbers multiply to give zero, then (at least) one of the numbers must equal zero.

***Example** 1.5.2* Understand each of the following:

$$0 \times 57 = 0 \qquad \frac{0}{7} = 0 \qquad \frac{0}{-1} = 0$$

$\frac{0}{0}$ is undefined, and $\frac{7}{0}$ is undefined.

1.6 Order of operations

- Consider the expression $2 + 3 \times 5$.
  - do we add 2 to 3, and then multiply by 5, giving 25; or
  - do we multiply 3 by 5, and then add 2, giving 17?
- We need rules for **order** of operations.
- The word BEDMAS (or BOMDAS, or BODMAS) can help you remember the rules.
- Each letter stands for a common mathematical operation; the **order** of the letters matches the **order** of doing the mathematical operations

letter	stands for:	example
B	brackets	$(3+4)$
E	exponentiation	3^4
D	division	$3/4$
M	multiplication	3×4
A	addition	$3+4$
S	subtraction	$3-4$

- The basic rule is **work from left to right**, with the exact order decided by BEDMAS.
  - B: Look for any brackets in the expression, and evaluate inside the brackets **first**. If there are brackets inside brackets, then the **innermost** brackets get evaluated **before** the outermost ones.
  - E: Next, any exponentiation must be evaluated.
  - D, M: Next, evaluate divisions and multiplications, working from **left to right**. **Note that even though D comes before M in BEDMAS, they have the same priority.** (The rule could also be called BEMDAS.)
  - A, S: Finally, evaluate any additions or subtractions, working **left to right**. **Even though A comes before S in BEDMAS, they have the same priority.**

***Example** 1.6.1* Evaluate $3+4\times 2$ and $(3+4)\times 2$.

(1) $3+4\times 2 = 3+8 = 11$

(2) $(3+4)\times 2 = 7\times 2 = 14$

***Question** 1.6.2* Evaluate each of the following expressions:

(1) $3 + 6 + 10 \div 5$

(2) $2 \times (1 + 4 \times (6 \div 3))$

(3) $12 \times 2 \div 3 \times 6 \div 12$

As a special case, if there are more than two **additions or multiplications all together (with no other operations)**, you can evaluate them in **any order**.

***Question** 1.6.3* Show that $2 \times 3 \times 4 = 3 \times 2 \times 4 = 4 \times 2 \times 3$.

1.7 Prime numbers and factors

- Given an integer, a second integer is called a *factor* of the first if it divides exactly into the first.
- If two or more integers share the same factor, then this is called a *common* factor of the integers.

***Example** 1.7.1* $14 = 2 \times 7 = 1 \times 14$. Hence the factors of 14 are $1, 2, 7$ and 14.

Finding factors

There are some simple tricks that let us easily check whether a small natural number is a factor of a given integer. A given integer is divisible by:

2	*if the last digit of the integer is even;*
3	*if the sum of the digits in the integer is divisible by 3;*
4	*if the last two digits form a integer divisible by 4;*
5	*if the integer ends in 0 or 5;*
6	*if the integer is divisible by both 2 and 3;*
9	*if the sum of the digits in the integer is divisible by 9;*
10	*if the integer ends in 0.*

***Example** 1.7.2*

- 295 is not divisible by 3, as $2 + 9 + 5 = 16$, then $1 + 6 = 7$, and 7 is not divisible by 3.
- 924 is divisible by 6, as 4 is even and $9 + 2 + 4 = 15$, which is divisible by 3.
- 94682128 is divisible by 4, as 28 is divisible by 4.
- 11875 is divisible by 5 but not divisible by 10.

- A **prime number** is a natural number, greater than 1, whose only factors are 1 and itself.

***Example** 1.7.3*

- 12 is NOT prime because $12 = 3 \times 4$; thus 3 and 4 are factors of 12.
- 17 is prime (check this using the tricks on the previous page).
- The first 7 prime numbers are 2, 3, 5, 7, 11, 13, 17.
- 2 is the only even prime number.
- By convention, 1 isn't prime.

- *Any* natural number larger than 1 is either prime, or can be written as a product of prime factors.

***Example** 1.7.4*

- 5 is prime.
- $21 = 3 \times 7$, and $3, 7$ are prime.
- $8 = 4 \times 2 = 2 \times 2 \times 2$.

***Question** 1.7.5* Write each of the following as the product of prime factors (if it's not already prime):

(1) 12

(2) 31

(3) 48

1.8 Fractions

- A fraction is the ratio of an integer (the *numerator*) *divided* by another integer (the *denominator*).
- Given a fraction $\frac{a}{b}$, its *inverse* is $\frac{b}{a}$

***Example** 1.8.1* The inverse of $\frac{2}{3}$ is $\frac{3}{2}$.

- Consider the fractions $\frac{1}{2}, \quad \frac{3}{6}, \quad \frac{4}{8}, \quad \frac{49}{98}$.
- Each of these fractions has exactly the same value: they are *equivalent* fractions.
- A fraction is written in *simplest form* or *lowest terms* if the numerator and denominator have no common factors other than 1. **In your final answer, you must always find the simplest form.**
- The process of converting fractions to simplest form is called *cancelling common factors.*
- To do this, we look for a common factor of the numerator and denominator, divide both by this factor; repeating until the only common factor is 1.
- (Cancelling works since anything divided by itself equals 1.)

***Example** 1.8.2*
$$\frac{20}{30} = \frac{2 \times 2 \times 5}{2 \times 3 \times 5} = \frac{2}{3}$$

(We obtained the final answer by cancelling $2 \times 5 = 10$ from the top and bottom of the fraction.)

- Soon we will see how to perform arithmetic on fractions.
- To do so, we often need to **convert** a fraction to an **equivalent fraction** with a **larger** denominator. (Remember, cancelling involves converting to an equivalent fraction with a *smaller* denominator.)
- This is done by multiplying **both** the numerator **and** denominator by the same quantity (so once again, we do the same thing to both the top and the bottom).

***Example** 1.8.3* Rewrite $\frac{1}{3}$ with a denominator of 12.

$$\frac{1}{3} = \frac{1 \times 4}{3 \times 4} = \frac{4}{12}$$

(Don't forget to check your answer by cancelling.)

- If two or more fractions have the same denominator then they are said to have a *common denominator*.

***Question** 1.8.4* Write $\frac{2}{3}$ and $\frac{3}{4}$ with a common denominator.

***Question** 1.8.5* Write $\frac{3}{8}$ and $\frac{1}{4}$ with a common denominator.

Arithmetic involving fractions

The following rules show how to perform arithmetic on fractions. You must always write your final answer in simplest form.

- **To multiply two fractions**, put the product of their numerators as the numerator of the result, and the product of their denominators as the denominator of the result.
- That is, multiply the top numbers together and multiply the bottom numbers together.

Example *1.8.6*

$$\frac{3}{4} \times \frac{2}{3} = \frac{3 \times 2}{4 \times 3} = \frac{6}{12} = \frac{1}{2}$$

- **To divide two fractions**, *multiply* the first fraction by the *inverse* of the second fraction.
- That is, change $\div$ to $\times$, flip the second fraction, and then multiply them.

Example *1.8.7*

$$\frac{3}{4} \div \frac{1}{8} = \frac{3}{4} \times \frac{8}{1} = \frac{3 \times 8}{4 \times 1} = \frac{24}{4} = 6$$

Question *1.8.8* Evaluate $\dfrac{1}{5} \times \dfrac{1}{4} \div \dfrac{2}{40}$.

- To add (or subtract) fractions with a **common** denominator, we add (or subtract) the numerators and place the result over the common denominator.

Example *1.8.9*

$$\frac{13}{32} + \frac{7}{32} - \frac{4}{32} = \frac{13+7-4}{32} = \frac{16}{32} = \frac{1}{2}$$

- To add (or subtract) fractions with **different** denominators, we **must** convert them to **equivalent** fractions with a **common denominator**, then we proceed as above.

Example *1.8.10*

$$\frac{1}{4} - \frac{1}{6} + \frac{1}{3} = \frac{3}{12} - \frac{2}{12} + \frac{4}{12} = \frac{3-2+4}{12} = \frac{5}{12}$$

Question *1.8.11* Evaluate each of the following:

(1) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8}$

(2) $\frac{5}{3} + \frac{4}{3} \div (6 \div 3)$

Question *1.8.12* There are some very common errors when dealing with fractions. Each of the following examples is **incorrect**. In each case, work out the correct answer. (Parts (b) and (c) are particularly common errors.)

(a) $\dfrac{1}{2} + \dfrac{3}{4} = \dfrac{1+3}{4+2} = \dfrac{4}{6}$

(b) $\dfrac{4+2}{2} = 4$

(c) $\dfrac{6+4}{1+4} = \dfrac{6}{1} = 6$

(d) $\dfrac{2}{5} \times 3 = \dfrac{2 \times 3}{5 \times 3} = \dfrac{6}{15}$

(e) $\dfrac{2}{2} = 0$

(Remember, these are all incorrect!)

1.9 Introduction to exponentiation

- We need to be familiar with **exponent** or **power** form:

$$\begin{aligned} 4 \times 4 &= 4^2 \\ 4 \times 4 \times 4 &= 4^3 \quad \ldots \text{and so on.} \\ 4 \times 4 \times 4 \times 4 &= 4^4 \end{aligned}$$

- This is what is meant by the `E` (*exponentiation*) in `BEDMAS`
- 4^2 is pronounced "4 squared", or "4 to the power of 2";
- 4^3 is pronounced "4 cubed", or "4 to the power of 3";
- In the expression 4^3, 4 is called the **base** and 3 is called the **power** or **index**.

Exponentiation and negative numbers

- On Page 14 we saw that:
  $(-\text{ve}) \times (+\text{ve}) = -\text{ve}$ and $(-\text{ve}) \times (-\text{ve}) = +\text{ve}$
- Think about what this means for exponentiation.
- When raising a $-$ve number to a power:
  - If the power is **even**, the answer is **positive**.
  - If the power is **odd**, the answer is **negative**.

***Example** 1.9.1*

$$\begin{aligned} (-1)^5 &= -1 \times -1 \times -1 \times -1 \times -1 \\ &= 1 \times -1 \times -1 \times -1 \\ &= -1 \times -1 \times -1 \\ &= 1 \times -1 \\ &= -1 \end{aligned}$$

***Example** 1.9.2* $(-2)^2 = -2 \times -2 = 4$
$(-2)^3 = -2 \times -2 \times -2 = 4 \times -2 = -8$

***Question** 1.9.3* Evaluate each of the following:

(1) $(-1)^{17}$

(2) $(-1)^{356}$

BEDMAS and exponentiation

From BEDMAS, we know that exponentiation has higher precedence than any operation besides brackets.

***Example** 1.9.4* Understand each of these examples:

1. $3 + 4^2 = 3 + 16 = 19$
2. $2 \times 3^2 = 2 \times 9 = 18$
3. $(2 \times 3)^2 = 6^2 = 36$
4. $3 + 2^{(2+1)} = 3 + 2^3 = 3 + 8 = 11$

***Question** 1.9.5* Evaluate each of the following:

(1) $4 - 2^2 + (-3)^3$

(2) $(-2)^2 - 2^{\left(\frac{1}{2} \div \frac{1}{6}\right)}$

1.10 Square roots

The previous section on exponentiation explained how to square numbers. The **square root** of a number is a non-negative (that means positive or zero) number that when multiplied by itself (or *squared*), gives the original number. For example, the square root of 4 is 2, the square root of 25 is 5 and the square root of 0.81 is 0.9. The symbol used to denote square root is $\sqrt{\ }$, so $\sqrt{4} = 2$ and $\sqrt{0.81} = 0.9$. While there are two numbers which when squared give the value 4, namely 2 and -2, only one of these is called the square root of 4, namely $\sqrt{4} = 2$.

Example *1.10.1* $9 = 3^2$, so $3 = \sqrt{9}$. $16 = 4^2$, so $4 = \sqrt{16}$. $\sqrt{\frac{1}{4}} = \frac{1}{2}$, since $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

It is easy to show that a negative number **does not** have a square root. Recall back in multiplication of numbers on Page 14 we saw that a negative number can only be obtained by multiplying a positive number and a negative number (or vice versa) together. As we saw above, the square root of a number is a non-negative number that when multiplied by **itself** (or *squared*), gives the original number. Positive $\times$ negative is not multiplying by itself, so therefore negative numbers do not have square roots. (Check this on your calculator.)

- We can find the square root of numbers that are not integers. (We will see how to in the coming chapters.)
- A square root does not have to be an integer.
- An integer whose square root *does* happen to be an integer is called a *square number*.

Example *1.10.2*

- The first 11 square numbers are:

$$0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.$$

Question *1.10.3* Evaluate each of the following:

1. $\sqrt{9}$
2. $\sqrt{36}$
3. $\sqrt{100}$
4. $-\sqrt{64}$
5. $\sqrt{0.04}$
6. $\sqrt{0.0009}$ (You may wish to use a calculator for these last two.)

NOTES

NOTES

2 Algebra

Why are we covering this material?

- Often, rather than dealing with specific numbers, we need to undertake operations in a more general manner.
- Hence we need to be able to manipulate expressions involving letters (rather than only numbers).
- Whatever degree you are doing, you'll almost certainly use this material extensively.
- Many students find that their algebra skills let them down throughout the rest of semester.
- Try to understand this stuff, and practise if you need to.
- **Topics in this section are**
  - Introduction to algebra.
  - Expanding and factorising.
  - Formulae.
  - Solving absolute values.
  - Intervals on the real line.
  - Solving inequalities
  - Square roots.
  - Powers and Exponents.

2.1 Introduction to algebra

- Until now, we've only dealt with numbers.
- Letters are often used in mathematics. Usually they are standard letters like x and y, but sometimes we use Greek letters (such as π, pronounced *pi*).
- Letters are used in the following ways:
  - as *numerical placeholders*; for example, 'find $3x - 2$ where $x = 4$', so x is replaced by its numerical value.
  - as *specific unknowns*; for example, 'find x where $4x+1 = 9$'.
  - as *generalised numbers* or *variables*; for example, 'for any positive number x, $\sqrt{x}\times\sqrt{x} = x$' or 'for any rectangle, $A = l\times b$'.
  - as *constants*; for example the letter π is always used to represent the ratio of the circumference of a circle to its diameter, which is the irrational number $3.141592\ldots$.
- Sometimes if we need many letters, or they are related, we use a *subscript* on them, where the subscript is an integer which is greater than or equal to 0.

Example *2.1.1* These variables represent different quantities:

$$x_1, \quad x_{10}, \quad y_1, \quad z_{100}, \quad z_0, \quad z_{523}, \quad x_{27}.$$

- An **algebraic expression** is a combination of numbers, letters and mathematical operations. Commonly, the name is shortened to **expression**.

Example *2.1.2* Here are some examples of expressions:

$$x^2 - x + 4 \qquad x_1 - x_2 + x_1x_2 \qquad \pi r^2$$

- Don't be too worried about letters, as operations on them are very similar to operations on numbers. For example:
  - BEDMAS applies.
  - powers of variables are the same as powers of numbers: for example, $x^3 = x \times x \times x$.
  - Fractions with letters in them act just like fractions which contain only numbers (so, for example, you can cancel letters in fractions in the same way that you cancel numbers).
  - letters can appear in square root signs.
- However, having letters does complicate calculations a bit; make sure you understand the following material.

Multiplication and division

- It is easy to evaluate 3×4 to give 12.
- However, we cannot evaluate $3\times x$ if we don't know what value x has.
- By convention, we write $3\times x$ as $3x$. The multiplication sign becomes invisible and the number, called the *coefficient*, is **always** written before the letter.
- Similarly, we can evaluate $12\div6$ to give 2, but we cannot evaluate $x\div4$. Instead, we write this as a fraction $\frac{x}{4}$, which is the same as $\frac{1}{4}\times x$ or $\frac{1}{4}x$.
- When multiplying and dividing letters, the same rules apply as when operating on numbers.
- These are shown in the following example.

Example *2.1.3* Rules for multiplying and dividing:

Rule	Example
$a\times b = ab = ba = b\times a$	$3\times 4 = 4\times 3 = 12$
$a\times(-b) = (-a)\times b = -ab$	$3\times -4 = -3\times 4 = -12$
$(-a)\times(-b) = ab$	$-3\times -4 = 3\times 4 = 12$
$\frac{a}{b}\times\frac{c}{d} = \frac{ac}{bd}$	$\frac{3}{4}\times\frac{1}{5} = \frac{3\times 1}{4\times 5} = \frac{3}{20}$
$a\div b = a\times\left(\frac{1}{b}\right) = \frac{a}{b}$	$10\div 7 = 10\times\frac{1}{7} = \frac{10}{7}$
$\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}$	$\frac{3}{-4} = -\frac{3}{4} = \frac{-3}{4}$
$\frac{-a}{-b} = \frac{a}{b}$	$\frac{-3}{-4} = \frac{3}{4}$

Example *2.1.4* We can do this two ways:

$$\begin{aligned} & \frac{2\times 6x}{z}\times\frac{2\times z}{3} \\ = & \frac{12x}{z}\times\frac{2z}{3} \\ = & \frac{24xz}{3z} \\ = & 8x \end{aligned} \qquad \text{or} \qquad \begin{aligned} & \frac{2\times 6x}{z}\times\frac{2\times z}{3} \\ = & 2\times 2x\times 2 \\ = & 8x \end{aligned}$$

Question *2.1.5* Simplify $\frac{8\times ab}{c}\div\left(\frac{18\times ab}{c}\right)$.

Like terms

- Pieces of an expression separated by the operations "+" or "−" are called **terms**. For example, in the expression $3xy + 4x + 2y$, the terms are $3xy$, $4x$ and $2y$.

Like terms

Terms are called **like** *if they have the* **same** *letters, each raised to the* **same** *powers (and only their coefficients can differ).*

Example *2.1.6* Each of the following lines contains like terms (but terms that are not on the same line are not like).

- $x,\ 7x,\ -2x,\ 8x/3$
- $x^2,\ 6x^2,\ -3x^2/2$
- $x^2y,\ -8x^2y,\ 0.1x^2y$
- $3xy^2,\ -xy^2,\ 0.1xyy$

- Usually, expressions are rearranged so that like terms are grouped together.

Simplifying expressions

- Given an expression, we will often need to *simplify* it, which means rewriting it in a simpler form.
- This is done by evaluating the mathematical operations as far as possible, using BEDMAS to determine the order of the operations.
- Once you have done BEDM (brackets, exponentiation, division and multiplication), rearrange the expression with like terms grouped together, and finally perform addition and subtraction of like terms.

Addition and subtraction

- Given two like terms, they are added (or subtracted) by adding (or subtracting) their coefficients, to give a single term with the same letter(s) and power(s).
- Note: terms are only like if they include the **same letters each raised to the same powers** in each term. For example, you cannot simplify $x^2 + x^3$.

***Example** 2.1.7*

$$
\begin{array}{lrcl}
(1) & 2x + 3x & = & 5x \\
(2) & 3x + 2y + 4x & = & 7x + 2y \\
(3) & p^3 + 4p^2 + 17 - 3p^2 & = & p^3 + 4p^2 - 3p^2 + 17 \\
 & & = & p^3 + p^2 + 17 \\
(4) & 6x - x \times x + x^2 - 2x \times 3 & = & 6x - x^2 + x^2 - 6x \\
 & & = & 6x - 6x - x^2 + x^2 \\
 & & = & 0
\end{array}
$$

***Question** 2.1.8* Simplify each of the following:

(a) $2x + 4y \times 3x + 3x - y$

(b) $6x \div 2 + \sqrt{9} - 2x - \sqrt{16}$

2.2 Expanding and factorising

- From BEDMAS, we know that brackets must be evaluated first. This is easy if the brackets contain like terms.
- If not, you need to use some new techniques to remove the brackets (but still remaining consistent with the BEDMAS rule).

***Example** 2.2.1* Simplify $2(3x + 4x)$ and $2(3x + 2y)$.

In each case, BEDMAS indicates that the section inside brackets must be evaluated first. Then $2(3x + 4x) = 2(7x) = 14x$.

However, when evaluating $2(3x + 2y)$, the section inside the brackets does not contain any like terms, so cannot immediately be simplified.

A different approach, called *expanding*, resolves this problem. We can see how the approach arises by thinking about operations on numbers.

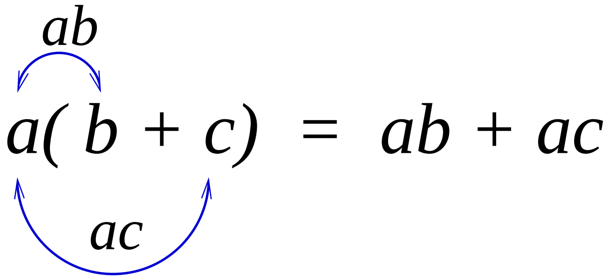
***Example** 2.2.2* Evaluate $2(3 + 6)$.

BEDMAS says that we must first calculate $3+6 = 9$, and then multiply by 2, giving $2\times9 = 18$.

If you think about this, you'll notice that when we multiplied 9 by 2, we were effectively multiplying **both** 3 **and** 6 by 2, and adding the answers. That is:

$$2(3 + 6) = 2\times3 + 2\times6 = 6 + 12 = 18.$$

This is demonstrated in the following diagram, in which we have written a, b and c instead of 2, 3 and 6.



- When expanding an expression like $a(b+c)$, first multiply the thing outside the brackets by **each of** the things inside the brackets, and then add your answers.
- Be careful with "−" signs and negative numbers.

Example *2.2.3*

- $3(p+2) = 3{\times}p + 3{\times}2 = 3p+6$
- $(c-4)d = c{\times}d - 4{\times}d = cd - 4d$

Question *2.2.4* Expand each of the following:

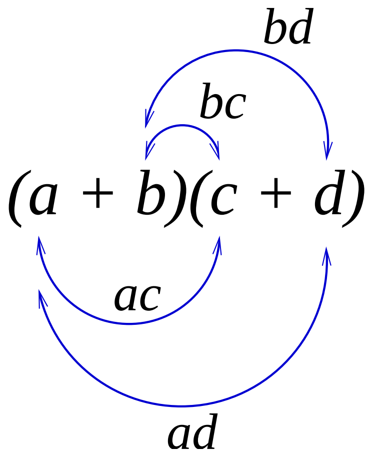
(a) $6(2-x)$

(b) $-3(y-4)$

(c) $-3(4x-y)$

(d) $-(x-1)$

- In more complicated examples there may be two sets of brackets multiplied together, such as $(4+5)\times(3+2)$.
- Again, it is easy to evaluate expressions which contain only numbers, but we need a different approach to expand similar expressions which contain letters.
- This is demonstrated in the following diagram, in which we have written a, b, c and d instead of 4, 5, 3 and 2.



$$(a+b)(c+d) = ac + ad + bc + bd$$

- When expanding something like $(a+b)(c+d)$, multiply **each of** the things inside the first brackets by **each of** the things inside the second brackets, and then add your answers.

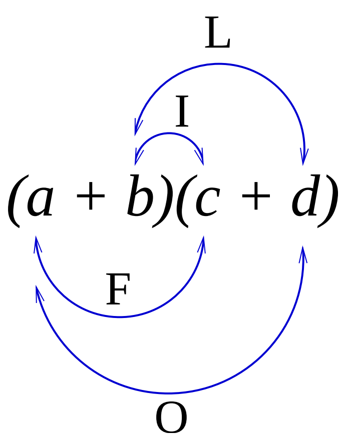
***Example** 2.2.5* Expand $(x+2)(y-3)$ and $(2x-3)(x-4)$.

$$\begin{aligned}(x+2)(y-3) &= x\times y + x\times(-3) + 2\times y + 2\times(-3)\\ &= xy - 3x + 2y - 6\end{aligned}$$

$$\begin{aligned}(2x-3)(x-4) &= 2x\times x + 2x\times(-4) - 3\times x - 3\times(-4)\\ &= 2x^2 - 8x - 3x + 12\\ &= 2x^2 - 11x + 12\end{aligned}$$

- Some people remember how to expand these brackets using the word FOIL, in which F stands for First, O for Outer, I for Inner and L for Last:

  Multiply the **first** terms of each brackets, then multiply the **outer** terms of each brackets, then the **inner** terms of each brackets and finally the **last** terms of each brackets.



- There are some special cases of these expanding rules, when the numbers in the first brackets are the same as those in the second brackets. Don't try to memorise them; use FOIL.

Example *2.2.6* Understand each of the following examples; pay particular attention to the "−" signs.

$$\begin{aligned}(a+5)(a+5) &= a^2+5a+5a+25\\ &= a^2+10a+25\\ (x-\sqrt{3})(x+\sqrt{3}) &= x^2+\sqrt{3}x-\sqrt{3}x-3\\ &= x^2-3\\ (a-3)(3-a) &= a\times 3-a\times a-3\times 3-3\times(-a)\\ &= 3a-a^2-9+3a\\ &= -a^2+6a-9\end{aligned}$$

Example *2.2.7*

$(a+b)\times(a+b) = a\times a + a\times b + b\times a + b\times b = a^2+2ab+b^2$

$(a-b)\times(a-b) = a\times a - a\times b - b\times a + b\times b = a^2-2ab+b^2$

$(a+b)\times(a-b) = a\times a - a\times b + b\times a - b\times b = a^2-b^2$

Question *2.2.8* Expand and simplify $(x-2)\times(3x-4)$.

Question *2.2.9* Expand and simplify $(2x-3)^2$.

Factorisation

- In previous problems, we have *expanded* expressions which contain brackets.
- For example, $6x(x-2) = 6x^2 - 12x$.
- Often the reverse process is useful, particularly in simplifying fractions and solving equations.

Factorising

Factorising *an expression involves finding a common factor in some or all of the terms of the expression, then rewriting the expression with the common factor multiplied by a new expression in brackets.*

Factorising is often harder than expanding. It takes skill and experience to identify common factors, and some expressions don't factorise.

When factorising, we usually use the "biggest" common factor.

***Example** 2.2.10* In each of the following, the expression on the left of "=" factorises to give the expression on the right. In each case, you should check the answer by expanding.

- $2x + 6 = 2(x+3)$
- $x^2 + 3x = x(x+3)$
- $-4n - 8n^2 = -4n(1+2n)$
- $36xy + 16xyz = 4xy(9+4z)$ (There are several common factors; we used $4xy$ as it is the "biggest" one.)
- $2p^2 + 2p - 8pqr = 2p(p+1-4qr)$

Question *2.2.11* Factorise each of the following expressions:

(1) $6st + 10s$

(2) $\dfrac{-3x}{2} + 3xy$

(3) $3ef + 5gh$

- Factorising is essential for simplifying fractions.

Example *2.2.12* Simplify $\dfrac{36xy + 16xyz}{2x}$.

Answer: In Example 2.2.10, we used the "biggest" common factor $4xy$ to factorise the numerator. When simplifying fractions, the denominator often gives a hint to a useful common factor in the numerator. In this example, because $2x$ is a common factor of the numerator, it will cancel with the denominator.

$$\frac{36xy + 16xyz}{2x} = \frac{2x(18y + 8yz)}{2x} = 18y + 8yz$$

Note that the final answer factorises to $2y(9+4z)$. Both answers are equally correct.

***Question** 2.2.13* Simplify each of the following:

(1) $$\frac{8x-8}{(x-1)(x+1)}$$

(2) $$\frac{6x+15xy}{12p+30py}$$

2.3 Equations and Formulae

- An **equation** relates letters and numbers to each other, using an equals sign.
- A **formula** (plural *formulae*) is an equation that gives a rule for calculating a particular quantity or thing

***Example** 2.3.1* Here are some examples of formulae:

1. $A = \pi r^2$
2. $P = A(1+r)^n$

- In each of these formulae, a value on the *left-hand side* (LHS) of the equals sign is given by an expression on the *right-hand side* (RHS) of the equals sign.
- We often choose letters carefully, to help the reader. For example, A often means area, V means volume, l means length, t means time, and so on.

Substituting into equations

- If values are known for all but one of the letters in an equation, then those values can be **substituted** into the equation, to enable the unknown value to be determined.
- Usually, the unknown is on the left-hand side of the equals sign, but that is not always the case.
- Often the equation needs to be determined from a "wordy" question, before values can be substituted into it.

***Question** 2.3.2* A rock is dropped vertically onto a lecturer's head, with initial speed $u = 4$. It accelerates at a rate of $g = 9.8$. The distance D it has travelled after t seconds is given by $D = ut + \frac{1}{2}gt^2$. How far has it travelled after 2 seconds?

Transposing equations

- Often we need to *rearrange* an equation, so that a particular letter occurs by itself on one side.
- Usually this is on the left, but sometimes it is easier to rearrange the equation with that letter on the right.
- This is often called **transposing** the equation.
- Note that we are **not** changing the equation, just writing it in a different order.

Rules for transposing equations.

1. *The same quantity may be added to, or subtracted from, both sides.*
2. *Both sides may be multiplied by, or divided by, the same quantity.*

- In words, this can be described as "whatever you do to one side, you must also do to the other side".
- Be careful: for example, you can't divide by zero.

***Example** 2.3.3* Transpose the following, to give $x = \ldots$

$$x - y = 4$$

To get x by itself, we need to move the $-y$ away from the left-hand side of the equation. We can do this by adding $+y$ to each side. Then:

$$\begin{aligned} x - y &= 4 \\ \text{so} \quad x - y + y &= 4 + y \\ \text{so} \quad x &= 4 + y \end{aligned}$$

- The equals sign in an equation signifies that the LHS is equal to the RHS, **and that the RHS equals the LHS**.
- For example, $x = 4$ is the same as $4 = x$.
- When transposing, you don't have to always isolate your variable on the LHS; sometimes it will be easier/nicer to move it to the RHS.
- Remember that you can swap the LHS and RHS at any time.

***Example** 2.3.4* Solve $x - 3 = 5x + 5$.

Answer:

We'll solve this in two ways. First, isolate x on the left:

$$\begin{array}{rrcl} & x-3 & = & 5x+5 \\ \text{so} & x-5x-3 & = & 5x+5-5x \\ \text{so} & -4x-3 & = & 5 \\ \text{so} & -4x-3+3 & = & 5+3 \\ \text{so} & -4x & = & 8 \\ \text{so} & x & = & -2 \end{array}$$

Alternately, isolate x on the right:

$$\begin{array}{rrcl} & x-3 & = & 5x+5 \\ \text{so} & x-3-x & = & 5x+5-x \\ \text{so} & -3 & = & 4x+5 \\ \text{so} & -3-5 & = & 4x+5-5 \\ \text{so} & -8 & = & 4x \\ \text{so} & -2 & = & x \\ \text{so} & x & = & -2 \end{array}$$

Regardless of which method we use, the answer is the same.

***Question** 2.3.5* Transpose each equation to give x by itself on one side of the equals sign, simplifying all like terms.

(1) $x - 3a = 0$

(2) $-x + 4y = x$

(3) $5 = \dfrac{y}{x}$

(4) $\dfrac{1}{x-3} = b$

Solutions to equations

- A **solution** to an equation is a set of values for each of the letters (i.e. variables) in the equation, which, when substituted into the equation, make the equation true.

***Example** 2.3.6* Show that $y = 3$ is a solution of the equation $y + 4 = 7$.

Answer: Substitute $y = 3$ into the left-hand side (LHS) of the equation, so $y + 4 = 3 + 4 = 7$, which matches the right-hand side (RHS).

Hence $y = 3$ is a solution.

***Question** 2.3.7* Show that $x = 6$ is a solution of the equation $2x - \dfrac{12}{x} = 10$.

- **Solving** an equation involves finding values for each of the letters so that the values give a solution to the equation.
- Usually you will need to rearrange the equation; writing a letter by itself on the left-hand side, and everything else on the right-hand side.
- A fundamental tool for solving equations is transposition; we covered this above.
- Remember: whatever you do to one side of the equation you *must* also do to the other side.

Question *2.3.8* Solve the equations:

(a) $3x + 4 = 2x$

(b) $2(x + 4) = 3(x - 2)$

(c) $24 - 2x = \frac{x}{2} + 4$

Check each answer by substituting it into the original equation!

2.4 Solving absolute values

- On Page 13 we encountered absolute values.
- The absolute value of x, written $|x|$, means the distance that x is from 0.
- Absolute value is always positive (or 0).
- By definition, $|x| = a$ means that $x = a$ or $x = -a$.

***Example** 2.4.1* If $|x| = 2$ then $x = 2$ or $x = -2$, which is sometimes written $x = \pm 2$.

- Thus, there are two values of x which satisfy $|x| = 2$.
- Similarly we can solve more complicated expressions which involve absolute value signs.
- In most cases, there will be two solutions to such problems.

***Example** 2.4.2* Solve $|x + 1| = 4$.

Answer: We are solving $|(\textit{something})| = 4$.

Hence we must have $(\textit{something}) = 4$ or $(\textit{something}) = -4$.

But here the $(\textit{something})$ is $x + 1$,

so either $x + 1 = 4$ or $x + 1 = -4$.

We solve each of these as a separate equation:

If $x + 1 = 4$ then $x = 4 - 1$, so $x = 3$.
If $x + 1 = -4$ then $x = -4 - 1$, so $x = -5$.

Hence the two solutions are $x = 3$ or $x = -5$.

(Check both answers by substituting each one into the original expression.)

Question *2.4.3* Find all x such that $|2x| = 6$.

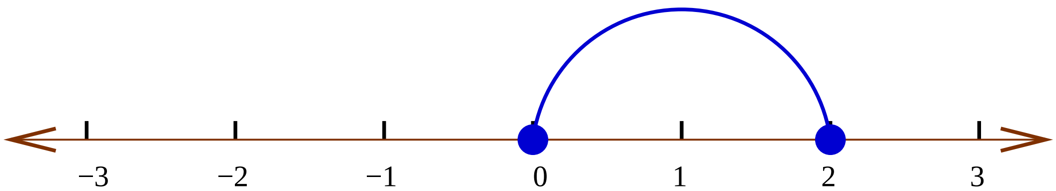
Question *2.4.4* Find all x such that $|4x + 2| = 2$.

Question *2.4.5* Find all x such that $|-2x - 3| = 7$.

2.5 Intervals on the real line

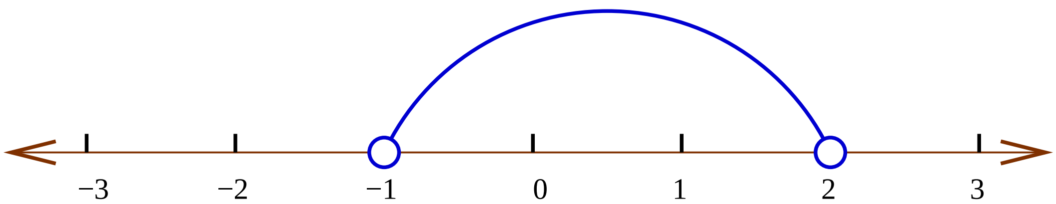
- On Page 12 we briefly encountered number lines (or *real* lines) and order (such as 'less than', written $<$).
- Any real number can be marked as a **single point** on the real line.
- **Intervals** or **regions** can also be marked on the real line. An interval includes *all real numbers which lie between two endpoints.*
- Such intervals can be described by *inequalities*, using the signs: $<$ $\leq$ $>$ $\geq$

***Example** 2.5.1* On the real line, mark the interval corresponding to $x \geq 0$ and $x \leq 2$.



We have highlighted the region between $x = 0$ and $x = 2$, with a solid black circle at each end point, and a (curved) line between the end points. This is used to denote **every point** between 0 and 2 (inclusive).

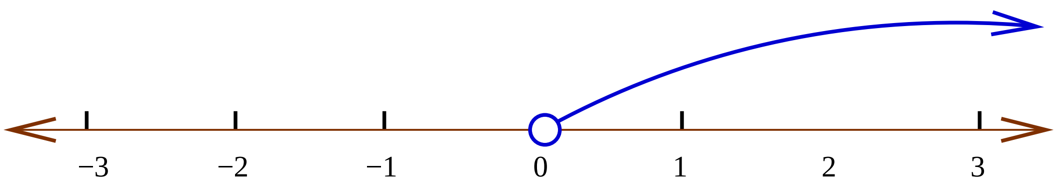
***Example** 2.5.2* On the real line, mark the interval corresponding to $x > -1$ and $x < 2$.



Now we have used a non-filled circle at each end point. This is used to denote **every point** between -1 and 2, but **not** including -1 and 2.

- Make sure you understand the difference between $\leq$ and $<$, and between $\geq$ and $>$
  - For $\leq$ and $\geq$ the endpoint occurs **inside** the interval, and is marked with a solid circle.
  - For $<$ and $>$ the endpoint occurs **outside** the interval, and is marked with a non-filled circle.
- Some intervals only have one endpoint (e.g. $x > 4$).
- This means that the interval goes on forever in one direction. If it goes to the right then we say it goes to infinity, written ∞. If it goes to the left, we say it goes to negative infinity, written $-\infty$.
- This is marked on a real line by an arrow pointing in the correct direction.

***Example** 2.5.3* On a real line, mark the region $x > 0$.



***Question** 2.5.4* Mark each of the following intervals on the real line:

(1) $x \leq 2$

(2) $-2 \leq x \leq 2$ (This means $-2 \leq x$ and $x \leq 2$.)

(3) $x < 2$

(4) $x > 2$

- There is an easy way to write intervals:
  - $[a, b]$ denotes the interval $a \leq x \leq b$
  - $[a, b)$ denotes the interval $a \leq x < b$
  - $(a, b]$ denotes the interval $a < x \leq b$
  - (a, b) denotes the interval $a < x < b$
- a and b are called the **endpoints** of the interval. Note that a (the first endpoint) is **always less than or equal to** b.
- Note the brackets: they indicate the type of interval.
  - A square bracket means the corresponding endpoint falls **inside** the interval. On the real line, the endpoint is marked with a solid circle.
  - A round bracket means the corresponding endpoint falls **outside** the interval. On the real line, the endpoint is marked with a non-filled circle. **(Note that $-\infty$ and ∞ always have a round bracket, not a square bracket.)**
- Be clear on what happens when an endpoint is **outside** an interval, e.g. $x > 0$. The point $x = 0$ is not in the interval, but every value greater than 0 is in the interval. So 0.5, 0.01, 0.000001 and 0.00000001 are all in the interval.

Question *2.5.5* Write each of the following intervals using inequality signs, and then mark each one on a real line:

(1) $(-\infty, 0)$

(2) $[0, 5)$

(3) $(0, 5]$

2.6 Solving inequalities

- We know how to solve equations with an "=" sign.
- The key rule was: whatever you do to one side, you must also do to the other side.
- We can also solve *inequalities*, which look like equations but instead have signs like $<$ or $\geq$.
- There are two major differences between equations and inequalities:
  - the answer to most inequalities is an **interval**, not a single point; and
  - the rules for manipulating inequalities are a bit different to those for solving equations.

Rules for solving inequalities.

1. *The same quantity can be added to, or subtracted from, both sides of the inequality.*
2. *Both sides of the inequality can be multiplied by, or divided by, the same* **positive** *quantity.*
3. *If both sides are* **multiplied** *by, or* **divided** *by, the same* **negative** *quantity, then the inequality must be reversed (that is, $<$ becomes $>$, $>$ becomes $<$, and so on).*
4. *If $a < b$ then $b > a$; if $a > b$ then $b < a$.*
   If $a \leq b$ then $b \geq a$; if $a \geq b$ then $b \leq a$.

- Rules 1 and 2 are the same as for solving equations.
- Pay particular attention to Rules 3 and 4: the inequality sign must be **reversed** when applying these rules!

***Example** 2.6.1* Solve the inequality $-3x+2 \le 6-x$.

$$-3x+2 \le 6-x$$

so $$-3x+2+x \le 6-x+x$$

so $$-2x+2-2 \le 6-2$$

so $$-2x \le 4$$

so $$-2x \div -2 \ge 4 \div -2 \quad \text{(the inequality is reversed)}$$

so $$x \ge -2$$

***Question** 2.6.2* Find all x which satisfy $2x-4 > x+3$. Write your answer in interval format and mark it on the real line.

***Question** 2.6.3* Find all x which satisfy $-2x \le x+3$. Write your answer in interval format and mark it on the real line.

***Question** 2.6.4* Find all y which satisfy $3(y+2) < 3y+4$.

2.7 Square roots

- We have previously seen square roots, written with a $\sqrt{\ }$ sign. If a is a real number then we know that:
  1. $\sqrt{a}$ is only defined if $a \geq 0$
  2. $\sqrt{a}\times\sqrt{a} = a$
  3. if $a > 0$ then a has two square roots, one positive and one negative.
- To avoid confusion, $\sqrt{a}$ is usually taken to mean the positive square root of a.
- To get the negative square root, write $-\sqrt{a}$.

The following rules allow us to simplify square roots.

Important properties of square roots.

If a and b are real numbers with $a \geq 0$ and $b \geq 0$, then

(1) $\sqrt{a} \times \sqrt{b} = \sqrt{a\times b} = \sqrt{ab}$

(2) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

***Example** 2.7.1*

1. $\sqrt{4} \times \sqrt{4} = \sqrt{4\times 4} = \sqrt{16} = 4$ and $\sqrt{7}\times\sqrt{7} = 7$
2. $\sqrt{5} \times \sqrt{20} = \sqrt{5\times 20} = \sqrt{100} = 10$
3. $\sqrt{\dfrac{4}{9}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}$
4. $-\sqrt{16} = -4$

***Question** 2.7.2* Simplify $\dfrac{\sqrt{8}\times\sqrt{6}}{\sqrt{16}}$.

?

- There are some common errors with square roots.
- Pay attention to the following facts; they each say that the two quantities are **not equal**.

Non-properties of square roots.

(1) $\sqrt{a}+\sqrt{b} \neq \sqrt{a+b}$

(2) $\sqrt{a}-\sqrt{b} \neq \sqrt{a-b}$

!

***Example** 2.7.3* Make sure you understand that:

$$\sqrt{2x}\times\sqrt{3y} = \sqrt{6xy}$$

but you cannot simplify:

$$\sqrt{2x}+\sqrt{3y}$$

√

***Question** 2.7.4* By letting $a = 9$ and $b = 16$, show that it is **not true** that $\sqrt{a}+\sqrt{b} = \sqrt{a+b}$.

?

Surds

- Some square roots can be written exactly as fractions; that is, they are **rational numbers**.

Example *2.7.5* The following square roots are rational:

$$\sqrt{4} = 2 = \frac{2}{1} \qquad \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$$

- Many square roots **cannot** be written exactly as fractions; that is, they are **irrational numbers**.
- For example, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$ are all irrational, and there is no way of writing them more simply.
- Irrational square roots are called **surds**.
- Sometimes, a surd can be written in a simpler form, by using the properties of square roots. In particular:

(1) $\sqrt{a^2} = a$ (for example, $\sqrt{16} = \sqrt{4^2} = 4$) and

(2) $\sqrt{a \times a} = a$ (for example, $\sqrt{2 \times 2} = 2$).

- These rules let us 'take things outside' the square root.

Simplifying square roots

Given a square root, we usually write it in simplest form by trying to 'take something outside' the square root sign. This is done via the following process:

- *Factor the number inside the square root sign, looking for*
  - *factors that are square numbers (e.g. $4, 9, 16, \ldots$); or*
  - *pairs of identical factors (if you don't easily find a square factor).*
- *Use rules (1) and (2) above to simplify.*

Example *2.7.6* Write $\sqrt{12}$ in simplest form.

Notice that 4 is a square number and that $12 = 4\times 3$. So:

$$\sqrt{12} = \sqrt{4\times 3} = \sqrt{4} \times \sqrt{3} = 2\times\sqrt{3} = 2\sqrt{3}$$

Alternatively,

$$\sqrt{12} = \sqrt{2\times 6} = \sqrt{2\times 2\times 3} = \sqrt{2\times 2} \times \sqrt{3} = 2\times\sqrt{3} = 2\sqrt{3}$$

Question *2.7.7* Simplify $\sqrt{20}$.

Arithmetic on surds

- Surds can be involved in expressions. For example, $3 + \sqrt{5}$ is an expression involving a surd.
- Mathematical operations (such as addition, multiplication and so on) can be performed on such expressions.
- Be careful to remember BEDMAS and the relevant properties of square roots.

Example *2.7.8*

$$\begin{aligned}(\sqrt{2} + 5) + (\sqrt{2} - 6) - \sqrt{2} &= \sqrt{2} + \sqrt{2} - \sqrt{2} + 5 - 6 \\ &= \sqrt{2} - 1\end{aligned}$$

Example *2.7.9*

$$
\begin{aligned}
& 3\sqrt{2}\times 5\sqrt{6} + 10\sqrt{3} \\
= \ & 15\times\sqrt{2}\times\sqrt{6} + 10\sqrt{3} \\
= \ & 15\sqrt{12} + 10\sqrt{3} \\
= \ & 15\times 2\sqrt{3} + 10\sqrt{3} \\
= \ & 30\sqrt{3} + 10\sqrt{3} \\
= \ & 40\sqrt{3}
\end{aligned}
$$

Question *2.7.10* Show that $\sqrt{2}+\sqrt{2}+\sqrt{2}=\sqrt{18}$.

Question *2.7.11* Simplify $(\sqrt{8}-\sqrt{2})(\sqrt{2}+\sqrt{6})$.

2.8 Powers and Exponents

- On Page 25 we briefly encountered *exponentiation*.
- For example, $3^2 = 3\times3$.
- In the expression 3^2, 3 is called the *base* and 2 is called the *power*, *exponent*, or *index*.
- There are various rules that allow us to simplify operations involving powers. You must be familiar with these rules.

Power Rule 1: Product of powers

If a, m and n are real numbers, then:

$$a^m \times a^n = a^{m+n}$$

Note that in this rule, the base must be the same in **both** *places on the LHS of the equals sign* **and** *on the RHS.*

Example *2.8.1*

- $2^2\times2^3 = 2^{2+3} = 2^5 = 32$
  You can see why the rule works:
  $2^2\times2^3 = (2\times2)\times(2\times2\times2) = 2\times2\times2\times2\times2 = 2^5$
- $y^3\times y^2\times y = y^{3+2+1} = y^6$
- We cannot simplify $x^3\times y^2$ as the first base x is not the same as the second base y. The most we can do is simplify it to x^3y^2.

***Question** 2.8.2* Simplify each of the following:

(a) $3^4 \times 3^2 \times 3^3$

(b) $x^7 \times x^2 \times y^4 \times x^6$

(c) $2^n \times 2^3$

Power Rule 2: Dividing powers

If a, m, n are real numbers, with a non-zero, then:

$$a^m \div a^n = a^{m-n}$$

Just as in Rule 1, the base must be the same in **both** *places on the LHS of the equals sign* **and** *on the RHS.*

***Example** 2.8.3*

- $3^5 \div 3^2 = 3^{5-2} = 3^3 = 27$

  You can see why the rule works:

  $3^5 \div 3^2 = \dfrac{3\times3\times3\times3\times3}{3\times3} = 3\times3\times3 = 3^3$
- $p^{10} \div p^6 = p^{10-6} = p^4$

***Question** 2.8.4* Simplify each of the following:

(a) $-7^{12} \div -7^5 \div -7^3$

(b) $x^4 \div x^{-4}$

(c) $3^{n+4} \div 3^{n+2}$

Power Rule 3: Power equal to 0 or 1

If a is any non-zero real number then:

$$a^0 = 1 \qquad \text{and} \qquad a^1 = a$$

Example *2.8.5*

- $3^2 \div 3^2 = 1$ and $3^2 \div 3^2 = 3^{2-2} = 3^0$. Demonstrating $3^0 = 1$.
- $x^4 \div x^3 = \dfrac{x \times x \times x \times x}{x \times x \times x} = x$ and $x^4 \div x^3 = x^{4-3} = x^1$.

  So it must be that $x^1 = x$.

Question *2.8.6* Simplify each of the following:

(a) $(2^{52} \times (-0.14536)^5)^0$

(b) $x^2 \times x \times x^3 \div x^5$

(c) $x^2 \times x^0 + x^3 \times y^0$

Power Rule 4: Negative power

Let a be any non-zero real number and m be any real number, then:

$$a^{-m} = \frac{1}{a^m}$$

Note that the expression a^{-m} has been rewritten as a fraction and the power is now 'positive' m.

Example *2.8.7*

- $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

  You can see why the rule works:

  $10^3 \div 10^5 = \frac{10\times10\times10}{10\times10\times10\times10\times10} = \frac{1}{10\times10} = \frac{1}{10^2}$.

  But $10^3 \div 10^5 = 10^{3-5} = 10^{-2}$.

  Hence $10^{-2} = \frac{1}{10^2}$.

- $x^{-3} = \frac{1}{x^3}$.

- $\frac{1}{5^{-2}} = \frac{5^2}{1} = 25$.

Question *2.8.8* Simplify each of the following:

(a) $2^{-1}\times 10$

(b) $7^{-2}\times 14$

(c) $x^5 \times \frac{1}{x^4}$

Power Rule 5: Fractional powers

Let a be a real number and m be a non-zero real number, then:

$$a^{1/m} = \sqrt[m]{a}$$

In particular, for $m = 2$ we have $a^{1/2} = \sqrt[2]{a} = \sqrt{a}$. (For some values of m there are restrictions on allowed values of a. For example, if $m = 2$ then a cannot be negative.)

Example *2.8.9*

- $9^{1/2} = \sqrt{9} = 3$.

  You can see why the rule works:

  $9^{1/2} \times 9^{1/2} = 9^{1/2+1/2} = 9^1 = 9$.

  Hence $9^{1/2}$ must be $\sqrt{9}$.
- $(x^{1/2})^2 = (\sqrt{x})^2 = x$.
- $7^{1/3} \times 7^{1/3} \times 7^{1/3} = 7^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 7^1 = 7$.

Question *2.8.10* Simplify each of the following:

(a) $(3^2 \times 4^{1/2})^{1/2}$

(b) $x^{-1/2} - \dfrac{\sqrt{x}}{x}$

Power Rule 6: Powers raised to powers

If a, b, m and n are real numbers ($b \neq 0$ in the fraction) then:

$$(a^m)^n = a^{mn}, \quad (ab)^n = a^n b^n \quad \text{and} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example *2.8.11*

- $(4^2)^3 = 4^{2\times 3} = 4^6$

  You can see why the rule works:

  $(4^2)^3 = 4^2 \times 4^2 \times 4^2 = 4^{2+2+2} = 4^{2\times 3} = 4^6$
- $(x^2 y)^2 = x^{2\times 2} y^{1\times 2} = x^4 y^2$

Question *2.8.12* Simplify each of the following:

(a) $(2^{1/2})^2$

(b) $(x^2y^{-4})^{-1/2}$

(c) $\left(\dfrac{x^2}{y^3}\right)^{-3}$

Question *2.8.13* Evaluate $\dfrac{2^3 \times 4^{1/2} \times 36^{1/2}}{81^{1/4}}$.

(Hint: $81^{1/4} = (81^{1/2})^{1/2}$.)

Question *2.8.14* Simplify $x^2y \times y^{-2} \times (xy)^{-1}$.

***Question** 2.8.15* Simplify $x^2 \div (x^2y^{-2}) \times (x^2y)^{-2}$.

Summary of the power laws.

Let a, b, m, and n be real numbers. Then:

(1) $a^m \times a^n = a^{m+n}$

(2) $a^m \div a^n = a^{m-n}$

(3) $a^0 = 1 \quad (a \neq 0)$ *and* $a^1 = a$

(4) $a^{-m} = \dfrac{1}{a^m} \quad (a \neq 0)$

(5) $a^{1/m} = \sqrt[m]{a} \quad (m \neq 0)$

(6) $(a^m)^n = a^{mn}, \quad (ab)^n = a^n b^n$ *and* $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

(Of course, $b \neq 0$ in the fraction.)

- Note that Rules (1) and (2) only work when the **base is the same**. We can simplify $x^2 \times x^3$ to give $x^{2+3} = x^5$, but we cannot simplify $x^2 \times y^3$.

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3 Σ-notation

Why are we covering this material?

- This section covers sigma notation.
- Sigma notation is a short-hand way of writing long expressions involving addition.
- The notation is very important in probability and statistics.
- Anyone doing any economics statistics will need to use sigma notation a lot.
- **Topics in this section are**
  - Introduction to sigma notation.
  - Expanding sums.
  - Reducing sums.
  - Applications of sigma notation.

3.1 Introduction to sigma notation

Consider the following expressions:

***Example** 3.1.1*

1. $1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20$

2. $x_1+x_2+x_3+x_4+x_5+\ldots+x_{498}+x_{499}+x_{500}$

- Each of these expressions involves adding together a large number of terms.
- People often want to write very long sums, and it can be time consuming to write them out in full.
- There is a shorthand way of writing such expressions, using the Greek letter 'capital sigma', written Σ.
- In general, **sigma** or **summation** notation is written

$$\sum_{lower}^{upper} expression$$

  where

  - *lower* gives the lower bound or *starting point* of the sum,
  - *upper* gives the upper bound or *ending point* of the sum; and
  - *expression* gives the thing to be added together.

- *Lower* usually looks like *variable=value.*
- *Upper* usually is an integer or letter.
- *Expression* usually involves the *variable* given in *lower*.

***Example** 3.1.2* Here are three examples of sigma notation:

Σ	*lower*	*upper*	*expression*
$\sum_{i=1}^{5} i$	$i = 1$	5	i
$\sum_{i=-2}^{2} 2i + 4$	$i = -2$	2	$2i + 4$
$\sum_{i=1}^{100} 2^i$	$i = 1$	100	2^i

- There are two things you need to be able to do with sigma notation:
  - Given an expression involving Σ, expand it into a sum.
  - Given an expanded sum, reduce it into an expression involving Σ.

3.2 Expanding sums

- The shorthand notation can be expanded into a sum via the following process.

Procedure for expanding a sum.

- *Let the* variable *in* lower *equal the* value *in* lower.
- *While the* value *of the* variable *is* $\leq$ upper*:*
  - *Take the given* expression *and replace each occurrence of* variable *with its current* value.
  - *Add the resulting* expression *to the expanded sum*
  - *Add 1 to* value *of* variable

- Note: when expanding a sigma expression, you **always** add 1 to the *variable* at each step.

Example 3.2.1 $\sum_{i=1}^{6} i^2$ means "The sum of i^2, from $i = 1$ to $i = 6$." To expand this sum:

First, $i = 1$, so $i^2 = 1^2$. Add 1 to i, so $i = 2$.
Then $i = 2$, so $i^2 = 2^2$. Add 1 to i, so $i = 3$.
Then $i = 3$, so $i^2 = 3^2$. Add 1 to i, so $i = 4$.
Then $i = 4$, so $i^2 = 4^2$. Add 1 to i, so $i = 5$.
Then $i = 5$, so $i^2 = 5^2$. Add 1 to i, so $i = 6$.
Then $i = 6$, so $i^2 = 6^2$. Add 1 to i, so $i = 7$.
Then $i = 7$, which is larger than *upper*, so stop.

Then we obtain the expanded sum by adding together all of the terms obtained above:

$$\sum_{i=1}^{6} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2.$$

And you can simplify this even further:

$$\sum_{i=1}^{6} i^2 = 1^2+2^2+3^2+4^2+5^2+6^2 = 1+4+9+16+25+36 = 91.$$

- So far, in all the examples involving sigma we have always used the letter i as our variable.
- But we don't have to! We can **use any letter we like**.

Example 3.2.2 We saw that $\sum_{i=1}^{6} i^2 = 91$. Changing i to a different letter does not change the answer. So:

$\sum_{j=1}^{6} j^2 = 1^2 + \cdots + 6^2 = 91$ and $\sum_{k=1}^{6} k^2 = 1^2 + \cdots + 6^2 = 91$.

Example 3.2.3 $\sum_{r=2}^{4} r(r+1) = 2(2+1) + 3(3+1) + 4(4+1)$
$= 2\times 3 + 3\times 4 + 4\times 5 = 6 + 12 + 20 = 38$.

Question 3.2.4 Expand and simplify:

(1) $\sum_{i=1}^{3} x_i$

(2) $\sum_{j=-2}^{2} j$

(3) $\sum_{i=1}^{4} 2 + 0i$

(4) $\sum_{i=1}^{4} 2$

Question 3.2.5 Solve $\sum_{i=1}^{2} (2i+1)x = 16$.

3.3 Reducing sums

When given an expanded sum, to reduce it into an expression involving Σ, we need to identify each of:

- the starting point (or lower bound) of the sum,
- the ending point (or upper bound) of the sum, and
- what each term in the sum has in common.

Note: sometimes the sum might not appear to have an upper bound (denoted " + ... " to mean the sum goes on forever). This is expressed in Σ-notation by an upper-bound of infinity ("∞").

Example *3.3.1* Consider the following expanded sum and corresponding sigma expression.

$$1+2+3+4+5+6+7+\dots \quad = \sum_{i=1}^{\infty} i$$

Question *3.3.2* Write each of the following in sigma notation:

(1) $y_1 + y_2 + y_3 + y_4 + y_5$

(2) $-1 + 0 + 1 + 2 + 3 + 4$

(3) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

(4) $5^2 + 6^2 + 7^2 + 8^2 + 9^2 + ...$

(5) $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^n}$

Always check your answer (by expanding the Σ)!

So far it has been easy to see the lower bound, upper bound and expression being summed. Sometimes it is more complicated.

***Example** 3.3.3* Consider a sum of even numbers:

$$2 + 4 + 6 + 8 + 10$$

As always, the variable must increase by 1 at each step. So we can rewrite the expanded sum as:

$$2\times1 + 2\times2 + 2\times3 + 2\times4 + 2\times5$$

Thus, in sigma notation, $2 + 4 + 6 + 8 + 10 = \sum_{i=1}^{5} 2i$

- Any even number can be written as 2×(something).
- Any odd number can be written as 2×(something)+1.

***Example** 3.3.4* $1 + 3 + 5 + 7 + 9 = \sum_{i=0}^{4} (2i + 1)$

***Question** 3.3.5* Write each of the following in sigma notation:

(1) $4 + 6 + 8 + 10 + 12 + 14 + ...$

(2) $3 + 5 + 7 + \ldots + 29 + 31$

(3) $10 + 20 + 30 + 40 + 50 + 60 + 70$

3.4 Applications of sigma notation.

Sigma is useful in many practical applications.

Example *3.4.1* Consider the problem of calculating how many ancestors you have in total, going back n generations, including yourself.

n	description of ancestor(s)	no. ancestors in this generation	total ancestors
0	you	$1 \ (= 2^0)$	1
1	parents	$2 \ (= 2^1)$	3
2	grandparents	$4 \ (= 2^2)$	7
3	great grandparents	$8 \ (= 2^3)$	15
4	great great grandparents	$16 \ (= 2^4)$	31

An expression for finding the total number of ancestors that you have (including yourself) is:

$$\sum_{i=0}^{n} 2^i$$

Question *3.4.2* It can be proved that:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Show that this formula is valid for $n = 1$ and for $n = 4$.

Given a collection of data values (e.g. heights or incomes), people often talk about the average or *mean* value of the data.

***Question** 3.4.3* Assume there are 5 pieces of data, labelled x_1, x_2, x_3, x_4 and x_5.

(1) Write an expression, using sigma, for finding the mean value of this data.

(2) Given the following table of values, find the mean.

x_1	x_2	x_3	x_4	x_5
3	6	5	1	10

(3) The mean mark for six students is 8. Five students receive the marks: $m_1 = 8, m_2 = 9, m_3 = 9, m_4 = 7$ and $m_5 = 10$. Find the missing mark.

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4 Straight lines and their graphs

Why are we covering this material?

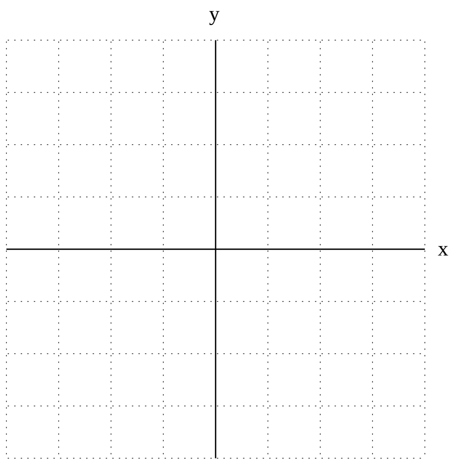
- This section covers several related topics: graph drawing, straight lines and distance.
- Graphs are very important for visualising the shape and behaviour of data sets and functions, and are used to illustrate relationships between various quantities.
- We will come back to graph drawing throughout semester, particularly for quadratics, exponentials and trigonometry.
- Linear (straight line) relationships are very common in representing and modelling real situations, especially in business and economics, engineering, genetics and biology.
- We will see how to graph linear relationships, and will study their general form and see how to recognise a linear relationship from its equation.
- We also study straight-line distance between two points and Pythagoras' Theorem.
- **Topics in this section are**
  - Introduction to graphs.
  - Sketching equations.
  - Linear graphs.
  - Standard form for the equations of straight lines.
  - Lines parallel to the axes.
  - Finding gradients.
  - Finding the equation of a line.
  - Parallel and perpendicular lines.
  - Measuring distance.

4.1 Introduction to graphs

- *Graphs* are used to show relationships between quantities. You should be familiar with the following aspects of graphs:
  - the horizontal axis (or x-axis) which is negative to the left and positive to the right;
  - the vertical axis (or y-axis) which is negative in the downwards direction and positive upwards;
  - the *origin*, where the axes intersect.
- Data values are written as **ordered** pairs or coordinates.
- The order of the values is important: for a point (a, b), x takes the value a and y takes the value b.

Question *4.1.1* Plot the following data points:

$$(x, y) = (2, 3), (1, 4), (0, -2), (-3, 4), (-1, -1)$$



- Often the variables being graphed are not x and y. For example, we may be looking at:

distance travelled	and	time taken; or
number	and	its square root; or
cost	and	number of pigs bought.

- Relabel the axes accordingly. For example, we might say, "time t is on the x-axis, and distance d is on the y-axis".

- The value of one variable **depends** on the value of the other one. The first variable is called the **dependent variable**, and the other is called the **independent variable**.
- By convention, the independent variable is **always** represented by the x-axis, and the dependent variable is always represented by the y-axis.
- It can sometimes be difficult identifying the dependent and independent variables, but something in the question will usually help you to work it out.
- For example, if a question asks you to plot the distance travelled against time spent travelling, distance is dependent on time.

4.2 Sketching equations

- You will often be asked to *sketch* (or *plot*, or *draw*) the graph of a given equation. Use the following procedure.

Sketching a graph.

Given an equation, sketch its graph by repeating these steps until you have enough points to be confident that you recognise the correct shape of the graph.

- *Choose a value for one of the variables x or y.*
- *Substitute the value of that variable into the equation, giving a value for the other variable.*
- *Plot each point (x, y) on some axes.*

Sketch the graph by joining appropriate points.

Example *4.2.1* Plot the graph of $y = x + 1$.

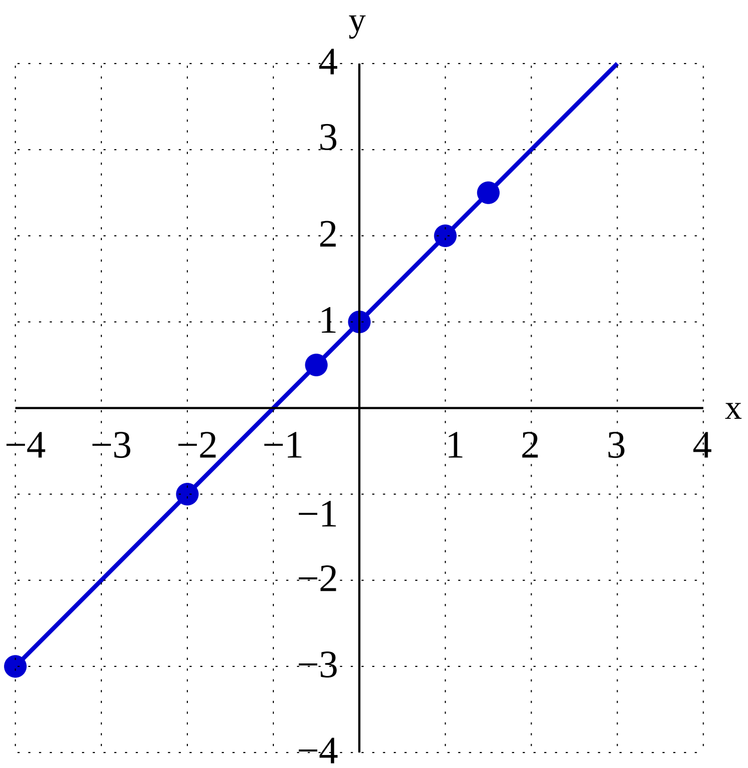
First we find some points on the graph, by choosing values for x and calculating values for y. (Note that the points plotted on a graph don't have to be integers; they can also be decimals or fractions. Usually when finding points we only use integers, for ease of calculation and plotting.)

$$\begin{array}{ll}
\text{When } x = 0, & y = x + 1 = 0 + 1 = 1 \\
\text{When } x = 1, & y = x + 1 = 1 + 1 = 2 \\
\text{When } x = -2, & y = x + 1 = -2 + 1 = -1 \\
\text{When } x = 1.5, & y = x + 1 = 1.5 + 1 = 2.5 \\
\text{When } x = -0.5, & y = x + 1 = -0.5 + 1 = 0.5 \\
\text{When } x = -4, & y = x + 1 = -4 + 1 = -3
\end{array}$$

It is often convenient to write these values in a table.

x	0	1	-2	2.5	-0.5	-4
y	1	2	-1	3.5	0.5	-3

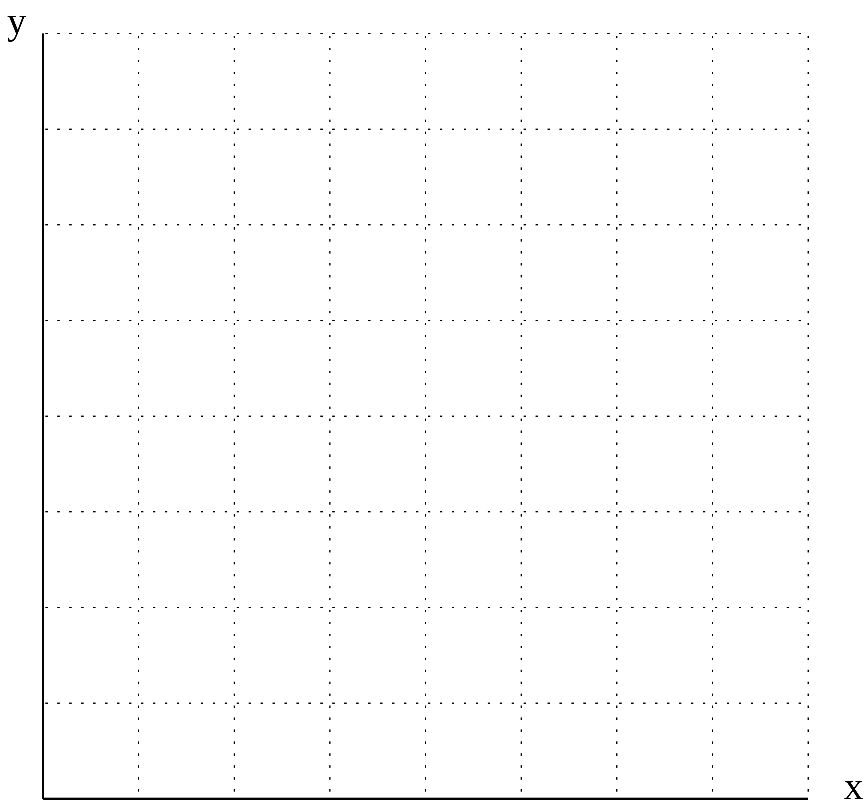
Finally we plot these points on a set of axes, and it's clear how the graph will look, so we don't need any more points.



- In the previous example, x and y could take on positive and negative values.
- Sometimes, particularly in practical problems, there might be some restrictions on values of x and y. For example, x might have to always be positive.

***Question** 4.2.2* A carwash costs \$10 and petrol costs \$1 per litre. Assume you buy x litres of petrol and have a car wash. Let y be the total cost. First create a table of values, then sketch a graph of y. Find an equation for y.

x	0	5	10	15	20	25
y						



The amount of money you spend is \$10 for the wash, plus x dollars for x litres of petrol. Hence the equation is: $y = 10 + x$.

4.3 Straight line (linear) graphs

- In Example 4.2.1 and Question 4.2.2 we calculated 6 points in order to plot the graph.
- The graphs each formed a straight line, and you might have noticed that we didn't need so many points to correctly identify the line.

Plotting a straight line.

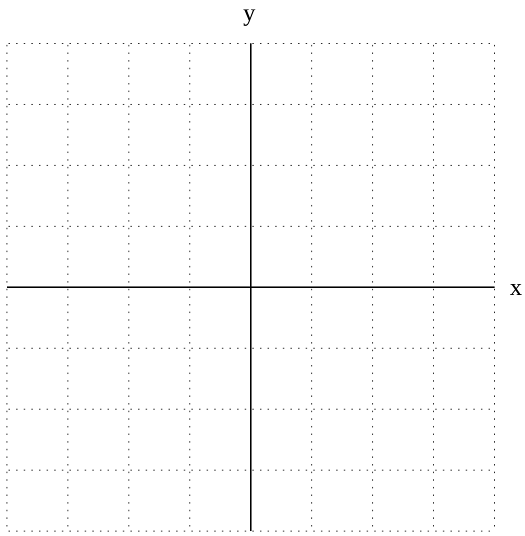
Given two points (x_1, y_1) *and* (x_2, y_2)*, there is a unique line which passes through both points.*

Equivalently, given the equation of a straight line, only two points on the line are needed in order to plot the graph.

- Thus, **if you know an equation is a straight line**, you only need to find two points on the line.
- You can choose **any** two points on the line (as long as they are different).
- Some points may be easier to calculate than other points.
- Common choices for the two points are:
  - The point where $x = 0$. This is called the *y-intercept*, and is the point where the graph crosses the y-axis.
  - The point where $y = 0$. This is called the *x-intercept*, and is the point where the graph crosses the x-axis.
- (Of course, you cannot use these point(s) if the values are not allowed; in Question 4.2.2, there was no point on the graph for which $y = 0$, as the total cost was at least \$10.)

***Question** 4.3.1* Use the x and y-intercepts to sketch the graph of $2x + 3y = 6$. (You may assume this is a straight line.)

x		
y		



- In Example 4.2.1, every additional litre of petrol purchased (the independent variable) added the same amount to the total cost (the dependent variable).
- We saw that the graph was a straight line. Another name for a straight line relationship is a **linear** relationship.
- In linear relationships, each time the independent variable changes by a certain fixed amount (say a), the dependent variable always changes by another fixed amount (say b).
- We saw above that if we know an equation is linear then we only need two points to plot its graph.
- There is an easy way to tell if an equation is linear.

Identifying whether a given equation is linear.

- *Simplify the equation as much as possible.*
- *The equation is linear if it contains* **at most***:*
  - *one term involving x (which may be zero)*
  - *one term involving y (which may be zero)*
  - *one constant term (which may be zero)*

- There can be no other terms; i.e. no terms involving xy, x^2, y^2, $\sqrt{x}$, etc.
- The highest power of x must be 1 (recall that $x^1 = x$) and the highest power of y must also be 1.
- As stated, some of these terms can be zero (in which case they will be missing).

Example *4.3.2* Which of the following equations are linear?

(a) $3x + 2y - 4 = x + 2$.

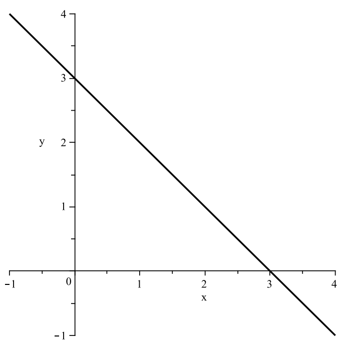
Simplifying the equation, we get $3x + 2y - 4 = x + 2$, so $2x + 2y - 6 = 0$. Hence there is a term involving x, a term involving y and a constant term, so the equation is linear.

(b) $3(x + y) = x(y + 1)$.

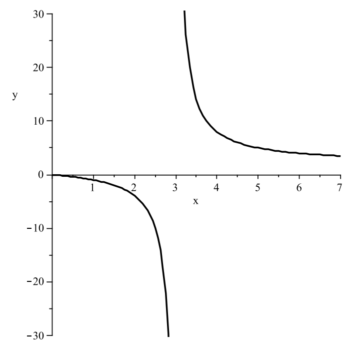
When we simplify the equation, we get $3x + 3y = xy + x$, so $2x + 3y = xy$, which includes a term involving xy. Hence this is not a linear equation.

(c) $3(x + 1) = y + 3$.

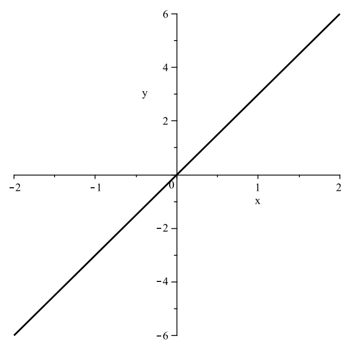
When we simplify the equation, we get $3x + 3 = y + 3$, so $3x = y$, which includes a term involving x and a term involving y (and no constant term). So the equation is linear.



Graph of (a)



Graph of (b)



Graph of (c)

***Question** 4.3.3* Which of the following equations are linear? If an equation is not linear, state why.

(1) $2x - 6y = 18$

(2) $x(2 + y) = 4$

(3) $2x + \sqrt{x} = 4 + y$

(4) $x = 4$

(5) $x = 2y(2 + y)$

(6) $y = 1$

(7) $(x + 1)^2 = y + x^2$

4.4 Standard form for the equations of straight lines

- We have just seen that the equation of a straight line has at most an x term, a y term and a constant term, some of which may be zero.
- There is a useful standard way of writing linear equations.

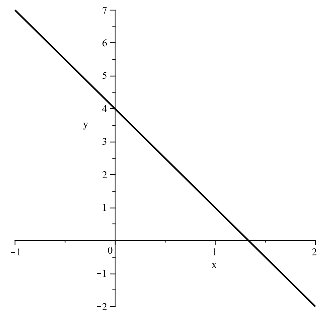
Writing straight line equations in standard form.

If a straight line equation has a y term then the standard way of writing the equation is: $y = mx + c$.

If there is no y term then the standard way of writing the equation is: $x = c$.

Example *4.4.1* Write the straight line $6x + 2y - 8 = 0$ in standard form, and identify the values of m and c.

We need to rewrite the equation as $y = mx + c$, where m and c are some constants.

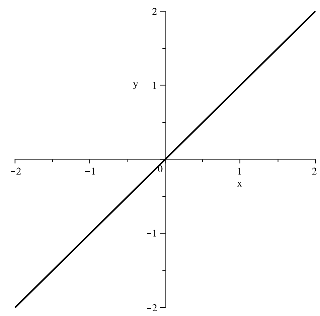


$$\begin{aligned} 6x + 2y - 8 &= 0 \\ \text{so} \qquad 2y &= -6x + 8 \\ \text{so} \qquad y &= -3x + 4 \end{aligned}$$

Hence $m = -3$ and $c = 4$.

Example *4.4.2* Write the straight line $2(y - 4) = 2x - 8$ in standard form, and identify the values of m and c.

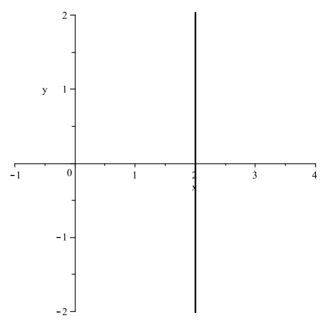
We need to rewrite the equation as $y = mx + c$, where m and c are some constants.



$$\begin{aligned} 2(y-4) &= 2x - 8 \\ \text{so} \qquad 2y - 8 &= 2x - 8 \\ \text{so} \qquad 2y &= 2x \\ \text{so} \qquad y &= x \end{aligned}$$

Hence $m = 1$ and $c = 0$.

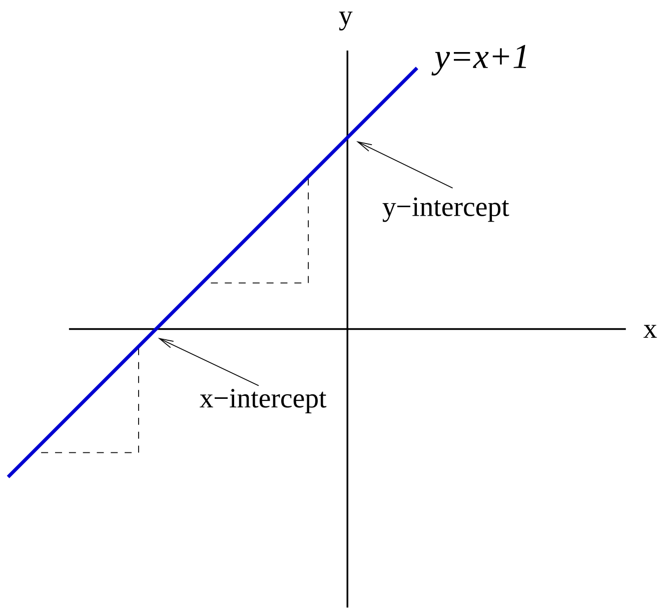
Example *4.4.3* Write the straight line $2(x + y) = 2y + 4$ in standard form.



$$\begin{aligned} 2(x+y) &= 2y + 4 \\ \text{so} \qquad 2x + 2y &= 2y + 4 \\ \text{so} \qquad 2x &= 4 \\ \text{so} \qquad x &= 2 \end{aligned}$$

(Note that here there is no y term.)

- There are three important features of straight line equations and their graphs:
  - The **y-intercept** of the equation is the value of y when $x = 0$. On the graph, it is value of y when the line crosses the y-axis.
  - The **x-intercept** of the equation is the value of x when $y = 0$. On the graph, it is the value of x when the line crosses the x-axis.
  - The **gradient** of the graph is its slope.
- These features are shown on the following diagram.
- The two triangles on that figure (in dashed lines) represent the slope or gradient of the line.



- Slope (of a line, or a road, or a hill) is a measurement of how quickly or steeply the height is changing.
- Formally, the **slope** of a line is the change in its y value between two points, divided by its change in x value between the two points.
- (Sometimes this is called "rise over run".)

- There are important relationships between the equation of a straight line in standard form and the graph of the line.

Gradient, y-intercept and standard form.

If a straight line is written in the form $y = mx + c$, then the gradient of the line equals m and its y-intercept equals c.

- It's not hard to see why those relationships hold.
- The y-intercept of the equation is the value of y when $x = 0$.
  - Let $y = mx + c$.
  - Substitute $x = 0$ into the equation, giving $y = m\times 0 + c$, so $y = c$.
  - Hence the y-intercept is c.
- The gradient is a measure of how much y changes compared to how much x changes.
  - Let $y = mx + c$.
  - In this equation, if x changes by a certain amount then y changes by m times that amount.
  - Hence the gradient is m.
- Now we know that m is the gradient, we see that:

Interpreting gradients.

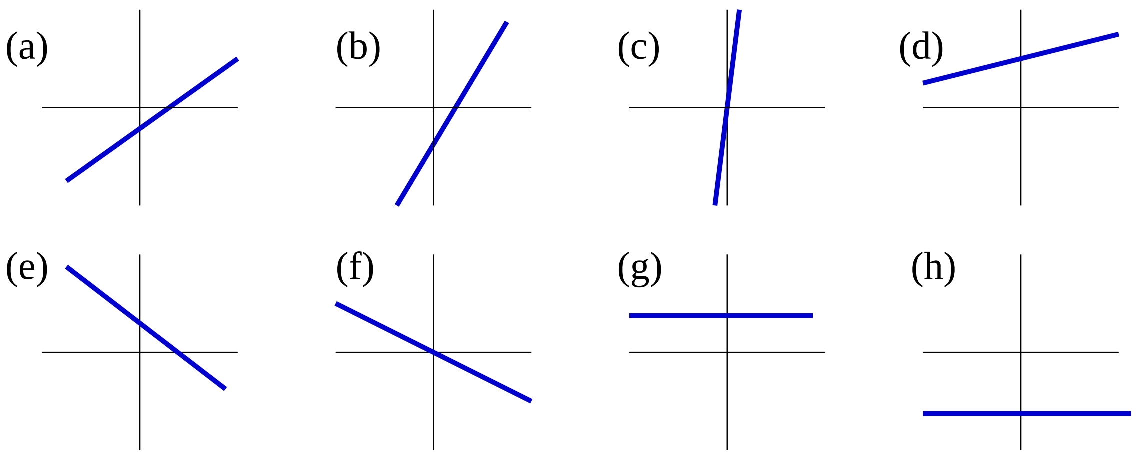
Let $y = mx + c$.

- *If m is* **positive**, *then as x gets bigger, y must increase. Hence the line goes* **upwards to the right**;
- *If m is* **negative**, *then as x gets bigger, y must decrease. Hence the line goes* **downwards to the right**; *and*
- *If m equals zero, then as x gets bigger, y does not change. Hence the line is* **horizontal**.

- Given a straight line equation in standard form it's easy to quickly find the gradient and y-intercept, so it's very easy to get a rough idea of what the graph looks like.
- If m is positive the line goes upwards to the right, and for larger values of m, will go up more steeply.
- If m is negative the line goes downwards to the right, and larger negative values of m, will go down more steeply.
- If c is:
  - positive, the graph crosses the y-axis **above** the x-axis.
  - negative, the graph crosses the y-axis **below** the x-axis.
  - zero, the graph crosses the y-axis **at the origin**.

Example *4.4.4* Draw a rough sketch of each of the following:

(a) $y = 2x - 4$ Hence $m = 2$ and $c = -4$.
(b) $y = 4x - 4$ Hence $m = 4$ and $c = -4$.
(c) $y = 8x$ Hence $m = 8$ and $c = 0$.
(d) $y = 2x + 4$ Hence $m = 2$ and $c = 4$.
(e) $y = -2x + 4$ Hence $m = -2$ and $c = 4$.
(f) $y = -x$ Hence $m = -1$ and $c = 0$.
(g) $y = 3$ Hence $m = 0$ and $c = 3$.
(g) $y = -4$ Hence $m = 0$ and $c = -4$.



4.5 Lines parallel to the axes

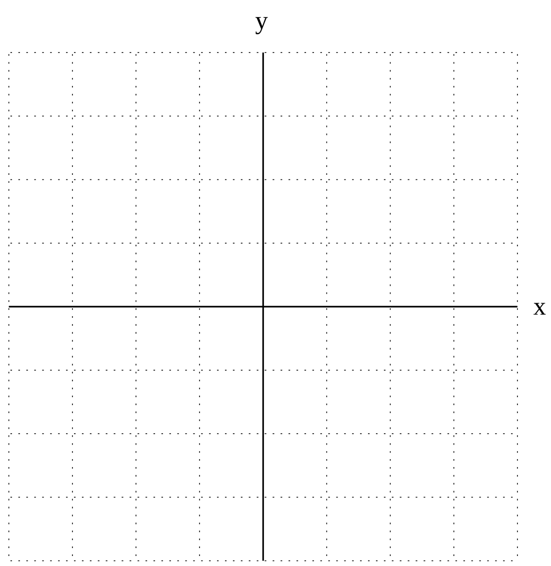
- In Example 4.4.4, the last two graphs were horizontal, or parallel to the x-axis.
- There is another special type of line: those which are parallel to the y-axis, and so are vertical.
- Such lines do not have equations that look like $y = mx + c$. Their general form is $x = c$.

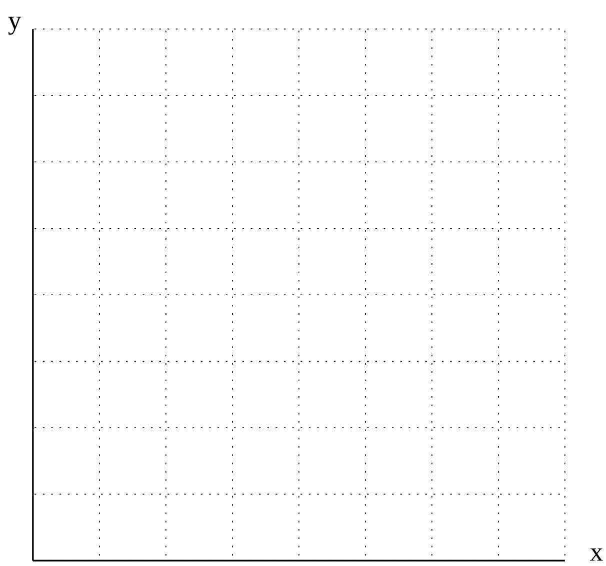
Lines parallel to the axes.

- *Horizontal lines have equations of the form $y = c$, where c is a constant. All such lines have gradient equal to 0.*
- *Vertical lines have equations of the form $x = c$, where c is a constant. All such lines have no gradient (this does not mean gradient equal to 0, it means* **no** *gradient), or sometimes they are said to have infinite gradient.*

***Question** 4.5.1* Sketch $y = 5$ for x between -4 and 4.

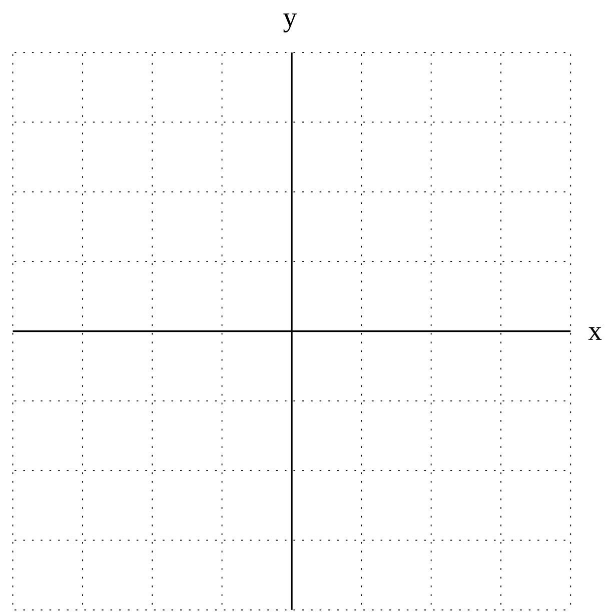
Sketch the cost y of eating in an all-you-can-eat restaurant against the amount you eat x, where the meal costs \$12.

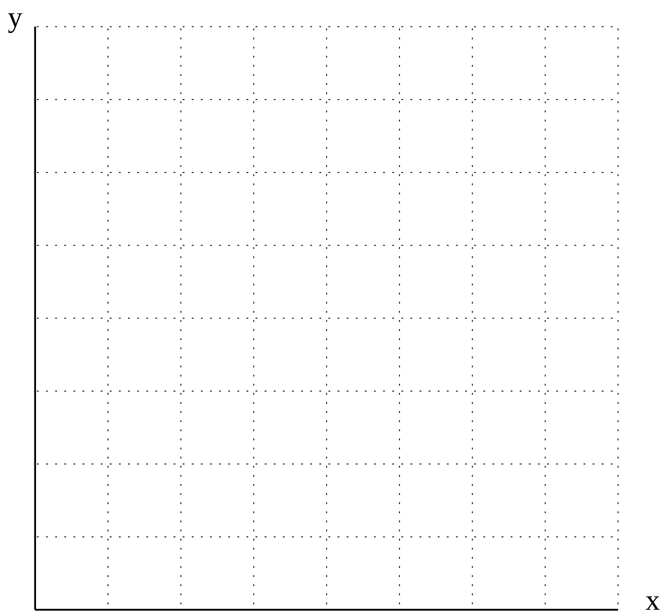




***Question** 4.5.2* **(a)** Sketch $x = -3$ for y between -8 and 8.

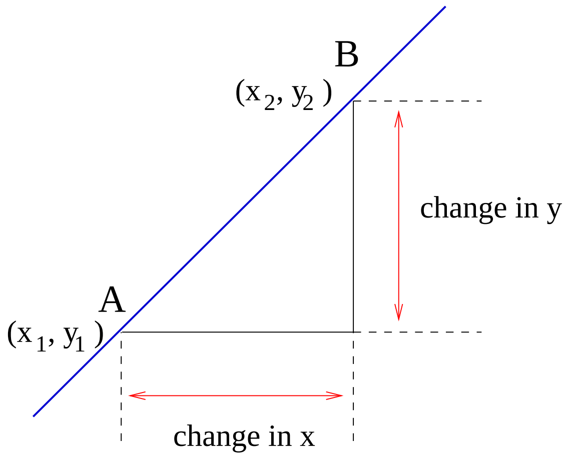
(b) When hiring a car for \$90 for a day, you can drive an unlimited distance without paying an additional fee. Sketch a graph of the distance you travel y as a function of the amount you spend on car hire x.





4.6 Finding gradients

- Given two points, it is often useful to find the gradient of the line passing through those points.
- (Of course, we have already seen that if the line is vertical then its gradient is undefined.)
- Let the two points be $A = (x_1, y_1)$ and $B = (x_2, y_2)$.
- The gradient of the line passing through A and B is the change in y values between A and B, divided by the change in x values between A and B.
- This is shown in the diagram on the next page.



- The y-coordinate of B is y_2 and the y-coordinate of A is y_1, so the change in y values between A and B equals $y_2 - y_1$.
- The x-coordinate of B is x_2 and the x-coordinate of A is x_1, so the change in x values between A and B equals $x_2 - x_1$.

Finding the gradient.

Given two points (x_1, y_1) and (x_2, y_2), the gradient of the line passing through these points is given by:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

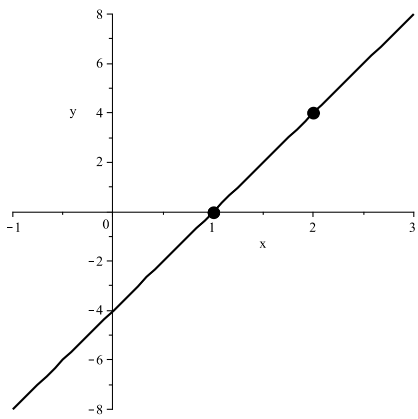
***Example** 4.6.1* Find the gradient of the line passing through the points $(1, 0)$ and $(2, 4)$. Show that it doesn't matter which point is chosen as (x_1, y_1) and which as (x_2, y_2).

Let $(x_1, y_1) = (1, 0)$ and $(x_2, y_2) = (2, 4)$.
Then:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{2 - 1} = \frac{4}{1} = 4.$$

Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (1, 0)$.
Then:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 4}{1 - 2} = \frac{-4}{-1} = 4.$$

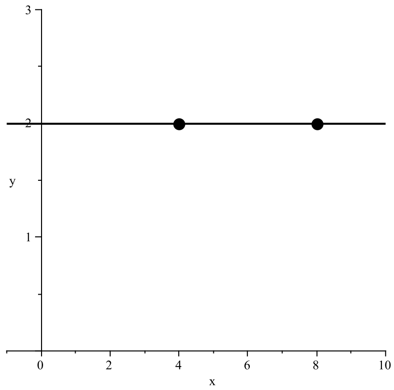


Example *4.6.2* Find the gradient of the line passing through the points $(4, 2)$ and $(8, 2)$.

Let $(x_1, y_1) = (4, 2)$ and $(x_2, y_2) = (8, 2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-2}{8-4} = \frac{0}{4} = 0.$$

Therefore the gradient is 0, so the line passing through $(4, 2)$ and $(8, 2)$ is horizontal.



- In the previous example, the change in y values was 0; this meant the line was horizontal.
- However, if the change in x values was 0, then the gradient formula would be dividing by zero, which is not allowed.
- Think about the points in this case: they must form a vertical line.
- Therefore the gradient is **undefined**.

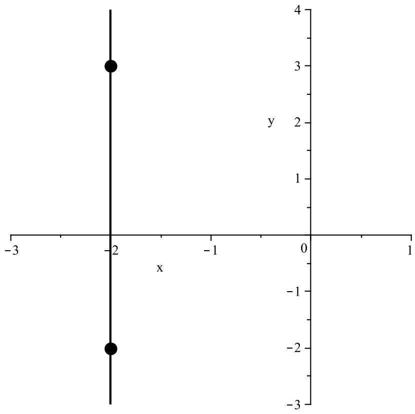
Example *4.6.3* Find the gradient of the line passing through the points $(-2, -2)$ and $(-2, 3)$.

Let $(x_1, y_1) = (-2, -2)$ and $(x_2, y_2) = (-2, 3)$.

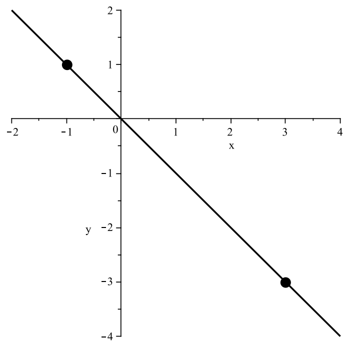
Then:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-2)}{-2 - (-2)} = \frac{5}{0}.$$

Therefore m is undefined. That is, the line passing through $(-2, -2)$ and $(-2, 3)$ is vertical.

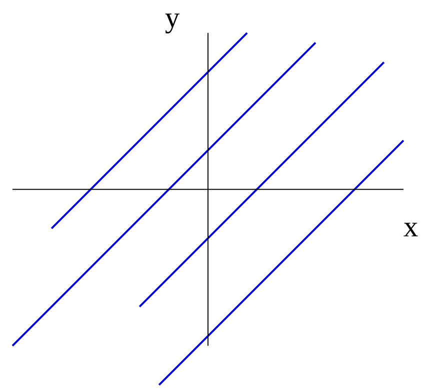


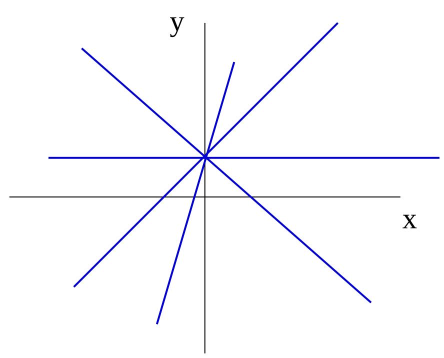
Question *4.6.4* Find the gradient of a straight line passing through the points $(-1, 1)$ and $(3, -3)$.



4.7 Finding the equation of a straight line

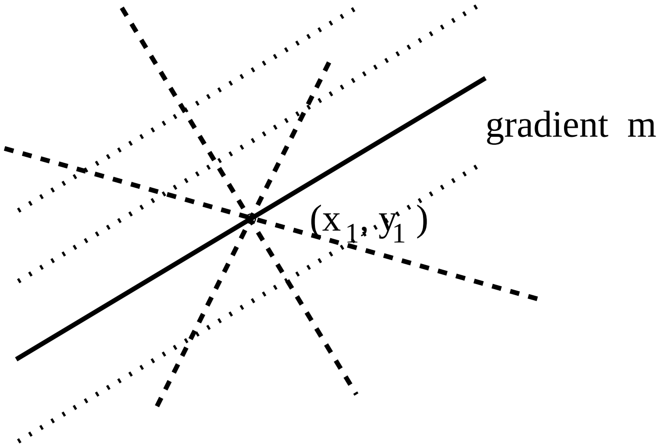
- There are a number of ways of finding the equation of a straight line, depending on what information is available.
- Note that you need to find both the gradient m and the y-intercept c.
- We can see what happens if just **one** of the gradient (left) or y-intercept (right) is known. In each case, an infinite number of lines is possible.





Finding the equation of the line when given a point on the line and the gradient of the line.

- If you are given a point on the line and the gradient of the line then this information defines a **unique** line.
- In the diagram: the lines with short dashes all have gradient m, and the lines with long dashes all pass through (x_1, y_1). Only the solid line has **both** properties.



Finding the equation given a point and the gradient.

To find the equation of a line given a point (x_1, y_1) on the line and the gradient m:

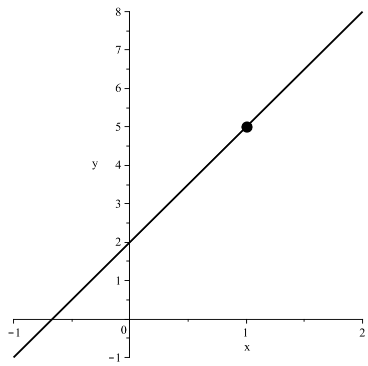
- *Write the value of m in the standard equation $y = mx + c$.*
- *Substitute the values of x_1 and y_1 into this equation and rearrange the equation, giving the value for c.*

***Example** 4.7.1* Find the equation of the line with gradient 3 which passes through the point $(1, 5)$.

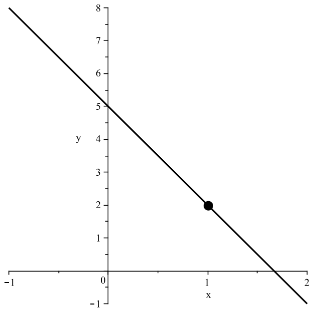
We know $m = 3$ and that $(1, 5)$ is on the line. The line has equation $y = 3x + c$, and hence $5 = 3\times 1 + c$, so $c = 2$.

Hence the equation of the line is:

$$y = 3x + 2.$$



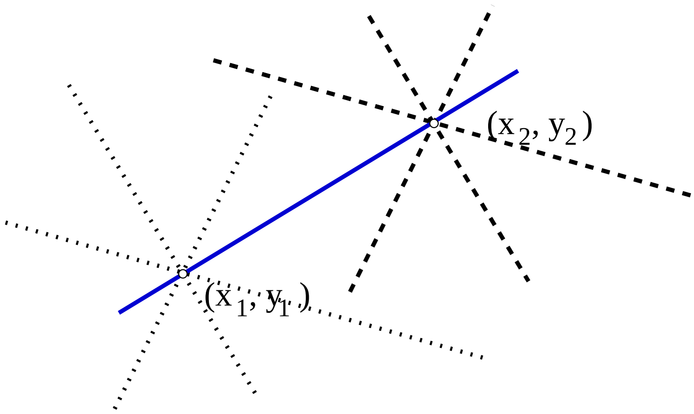
Question *4.7.2* Find the equation of the line whose gradient is -3, which passes through the point $(1, 2)$.



Question *4.7.3* Find the equation of the line with gradient 0, which passes through the point $(2, -2)$. Roughly sketch the graph.

Finding the equation of the line when given two points on the line.

- Assume you are given two points on the line.
- This information defines a **unique** line. The lines with short dashes all pass through (x_1, y_1) and the lines with long dashes all pass through (x_2, y_2). Only the solid line passes through **both** points.



Finding the equation given two points.

To find the equation of a line given two points (x_1, y_1) and (x_2, y_2) on the line:

- *Calculate the gradient using the formula:*
$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$
- *Write the value of m in the standard equation $y = mx + c$.*
- *Substitute the values of x_1 and y_1 (or x_2 and y_2) into this equation and rearrange, giving the value for c.*

- Of course, after the first step is done, we now know the gradient of the line and a point on the line (actually, we know two points, but we only need one).
- We then proceed exactly as before.

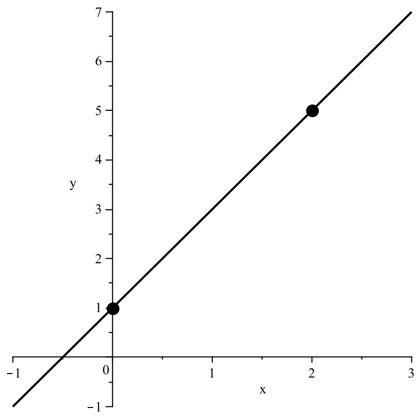
***Example** 4.7.4* Find the equation of the line which passes through the points $(0, 1)$ and $(2, 5)$.

Let $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (2, 5)$. Then:
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{2 - 0} = 2.$$

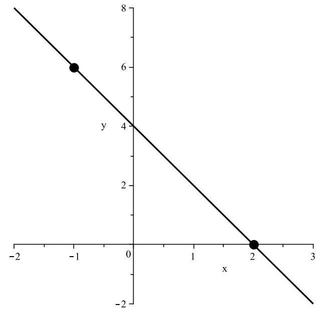
Then we proceed as before. Substituting for m gives $y = 2x + c$.

We must now choose either point to substitute into the equation. In this case $(0, 1)$ is likely to be easier. Hence:
$$1 = 2\times 0 + c, \text{ so } c = 1.$$

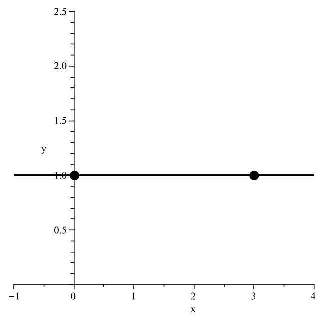
So the equation of the line is:
$$y = 2x + 1.$$



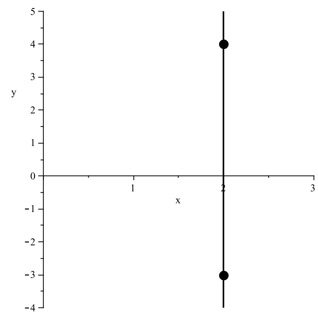
Question *4.7.5* Find the equation of the line which passes through $(2,0)$ and $(-1,6)$.



Question *4.7.6* Find the equation of the line which passes through $(0,1)$ and $(3,1)$.



Question *4.7.7* Find the equation of the line which passes through $(2,4)$ and $(2,-3)$.



Consider the last two questions! If both y values are equal, the equation will **always** be $y = constant$. If both x values are equal, the equation will **always** be $x = constant$.

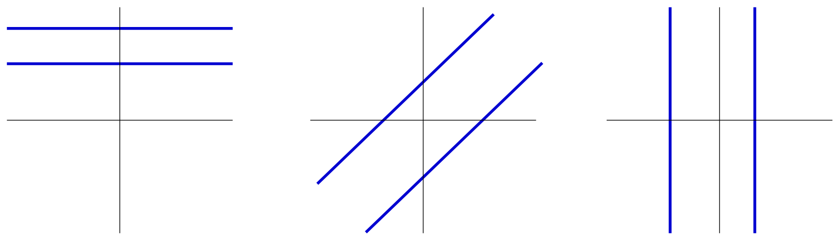
4.8 Parallel and perpendicular lines

- Given two lines, there are sometimes special relationships between their gradients.

Parallel lines.

Two lines are said to be parallel *if their gradients are equal or both are vertical.*

Example *4.8.1* Here are three pairs of parallel lines.

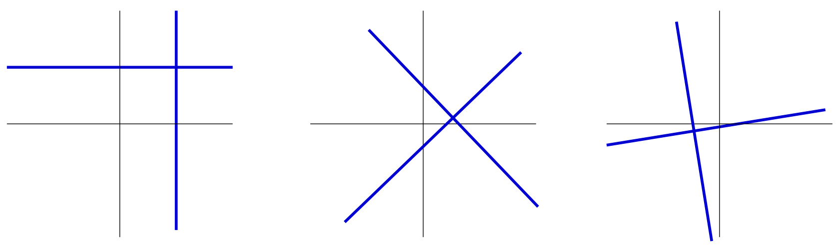


Perpendicular lines.

Two lines are said to be perpendicular *if they intersect in a right-angle. Either:*

- *one line must be parallel to the x-axis and the other to the y-axis; or*
- *neither line is parallel to the axes, and if one line has gradient m, the other line has gradient $-1/m$. (Another way of saying this is that the product of their gradients must equal -1.)*

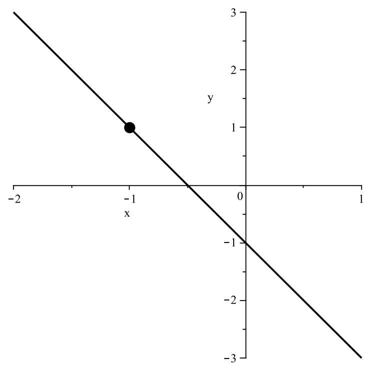
Example *4.8.2* Here are three pairs of perpendicular lines.



Example *4.8.3* Find the equation of the line parallel to $2y + 4x - 6 = 0$, passing through the point $(-1, 1)$.

Rewriting the equation in standard form gives $y = -2x + 3$, so the original line has gradient $m = -2$.

Hence any line parallel to this must also have gradient $m = -2$, so must have equation $y = -2x + c$.



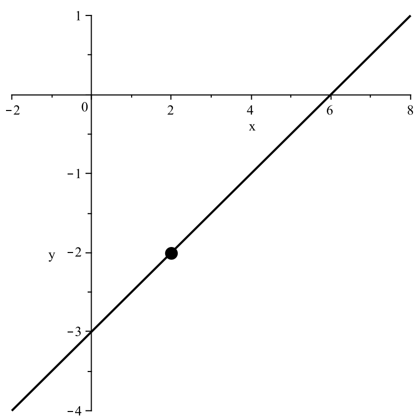
Substitute $(-1, 1)$ into the equation to find c. So $1 = -2\times -1 + c$, so $c = -1$.

Hence the equation is: $y = -2x - 1$.

Example *4.8.4* Find the gradient of any line perpendicular to $2y + 4x - 6 = 0$.

From Example 4.8.3 the original line has gradient $m = -2$, so any perpendicular line must have gradient $1/2$.

Question *4.8.5* Using Example 4.8.4, find the equation of the line perpendicular to $2y + 4x - 6 = 0$ and passing through the point $(2, -2)$.



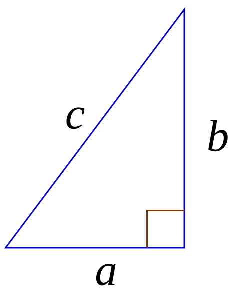
4.9 Measuring distance

- Next we cover a very important result: Pythagoras' Theorem.
- This is sometimes called the most important theorem in the whole of mathematics.
- It forms the entire basis of trigonometry and geometry.
- It is used to measure distances in 2 and 3 dimensions, so underpins engineering, architecture, geography and physics.
- It has been known for thousands of years.

Pythagoras' Theorem.

In a right-angled triangle with sides of length a and b and hypotenuse (i.e. the longest side) of length c, we have:

$$c^2 = a^2 + b^2 \quad \text{or equivalently } c = \sqrt{a^2 + b^2}$$



***Example** 4.9.1* In a right-angled triangle with hypotenuse of length 5 and another side of length 3, find the length of the remaining side.

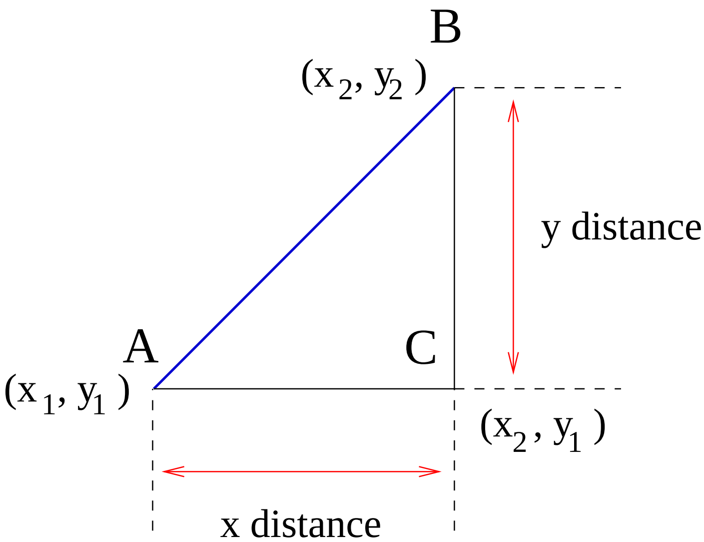
We have $c^2 = a^2 + b^2$, so $5^2 = 3^2 + b^2$, so $25 = 9 + b^2$, so $b^2 = 16$, so $b = 4$.

Hence the unknown side is of length 4.

Question *4.9.2* Wilfred starts walking at a point A.
He walks due east for $10\sqrt{2}$ km, then due north for $10\sqrt{7}$ km, ending up at point B. How far is B from A, in a straight line?

Distance between two points, (x_1, y_1) and (x_2, y_2).

- We can easily restate Pythagoras' Theorem to calculate the distance between any two points.
- Let the points be $A = (x_1, y_1)$ and $B = (x_2, y_2)$.
- Draw a triangle with A and B forming the endpoints of the hypotenuse, and find the lengths of the other sides.



- The vertical side BC has length equal to the difference in y values, which is $y_2 - y_1$.
- The horizontal side CA has length equal to the difference in x values, which is $x_2 - x_1$.

- From Pythagoras' Theorem, the length h of the hypotenuse can be found by noting that:

$$\begin{aligned} h^2 &= (CA)^2 + (BC)^2, \\ \text{so} \quad h^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2. \end{aligned}$$

Hence we have the following result:

Distance formula.

The distance d between two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- Of course, the distance from the point (x_1, y_1) to (x_2, y_2) must be the same as the distance from (x_2, y_2) to (x_1, y_1).
- This can be seen from the above distance formula.

***Example** 4.9.3* Find the distance between $(2, 3)$ and $(-2, 6)$.

Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-2, 6)$. Then:

$$\begin{aligned} d &= \sqrt{(2-(-2))^2 + (3-6)^2} = \sqrt{4^2 + (-3)^2} \\ &= \sqrt{16+9} = \sqrt{25} = 5. \end{aligned}$$

***Question** 4.9.4* Find the distance between $(\sqrt{6}, 3\sqrt{3})$ and $(0, \sqrt{3})$, expressing your answer as a surd in simplest form.

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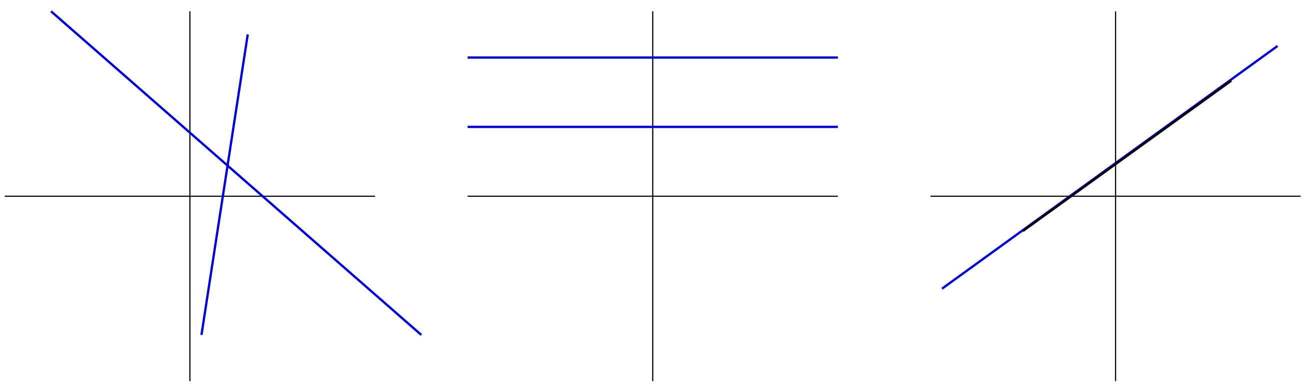
5 Intersecting lines; simultaneous equations

Why are we covering this material?

- This section looks at methods of finding the simultaneous solution(s) to a pair of equations.
- Each of the equations will be a straight line.
- Effectively, we are concerned with the possible intersection of a pair of lines.
- We will see two ways of finding the intersection: substitution and elimination.
- We only consider pairs of equations with two unknown variables. More generally, there can be many equations with many unknown variables.
- Computers were originally developed mostly for solving very large systems of equations.
- The techniques we learn can be applied to much larger systems.
- **Topics in this section are**
  - Intersection of lines.
  - Solving simultaneous equations.

5.1 Intersection of lines

- Consider two lines, $y = m_1x + c_1$ and $y = m_2x + c_2$.
- Every point on the first line satisfies the first equation, and every point on the second line satisfies the second equation.
- Here we are interested in any point(s) in common to **both** lines.
- Such points must satisfy both equations at the same time, so they are called *simultaneous solutions* to the equations.
- Any points which are in common to both lines will appear as points of intersection if the lines are drawn on a graph.
- If both lines are drawn on the same set of axes, then **one** of the following **must** happen:
  - the lines must intersect at precisely one point; or
  - the lines are parallel and do not intersect at all; or
  - the lines must intersect at an infinite number of points (so they must be superimposed, i.e. they are same line).
- These three possibilities are shown in the following figure. (In the right-most graph, two lines are superimposed.)



- It is not possible for straight lines to intersect in any other number of points (such as 2 points, or 20 points).

- There are many examples of simultaneous equations which need to be solved in everyday life.
- We will concentrate on problems with two equations, both of which are straight lines.
- In general there may be more than two equations, and they may not be straight lines.

Simultaneous solutions.

Given the equations of two lines:

- *If they intersect in precisely* **one point**, *say* (x_i, y_i), *then* (x_i, y_i) *satisfies* **both** *equations at the* **same** *time, and is called the* **simultaneous solution** *to the equations.*
- *If the lines* **do not intersect** *at all, then they must be parallel. We say that there is* **no simultaneous solution** *to the given equations.*
- *If the lines intersect in an infinite number of points, then they must be the* **same line**. *We say that there is an* **infinite number of simultaneous solutions**.

5.2 Solving simultaneous equations

- Given a set of equations, *solving them simultaneously* involves finding all simultaneous solutions to the equations.
- We will encounter two similar techniques, called *substitution* and *elimination*.
- When solving simultaneous equations, it is common to label the equations with numbers in brackets, such as (1).

Solving simultaneous equations using substitution.

- *Given two equations, choose one equation and isolate one of the variables to the left-hand side.*
- *Substitute the expression for that variable into the other equation (thus eliminating the identified variable from the other equation).*
- *Solve the resulting equation for the remaining variable.*
- *Substitute the value of that variable into either original equation, giving the value for the other variable.*

***Example** 5.2.1* Solve:

$$2x - y = -4 \quad (1)$$
$$x + 2y = 28 \quad (2)$$

Answer: Rewrite (2) with x on the left-hand side, so:

$$x = 28 - 2y \quad (3)$$

Now substitute for x in Equation (1), giving:

$$2(28 - 2y) - y = -4 \quad (4)$$

Now (4) is an equation only involving x which we can solve:

$$\begin{aligned} 2(28 - 2y) - y &= -4 \\ \Rightarrow \quad 56 - 4y - y &= -4 \\ \Rightarrow \quad -5y &= -60 \\ \Rightarrow \quad y &= 12 \end{aligned}$$

Thus we know that $y = 12$, so we can substitute $y = 12$ into either (1) or (2). Let's choose (1), so:

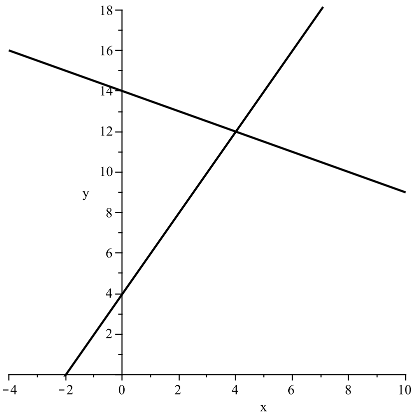
$$2x - y = -4 \quad \Rightarrow \quad 2x - 12 = -4 \quad \Rightarrow \quad x = 4$$

continued...

Example *5.2.1 (continued)*
Thus the solution is $x = 4$ and $y = 12$; that is, $(4, 12)$.

Finally, we can check that this is the correct solution, by substituting these values of x and y into both of the original equations (1) and (2).

From (1), $2x - y = -4$. Let $x = 4$ and $y = 12$, so $2x - y = 2(4) - 12 = -4$ as required.



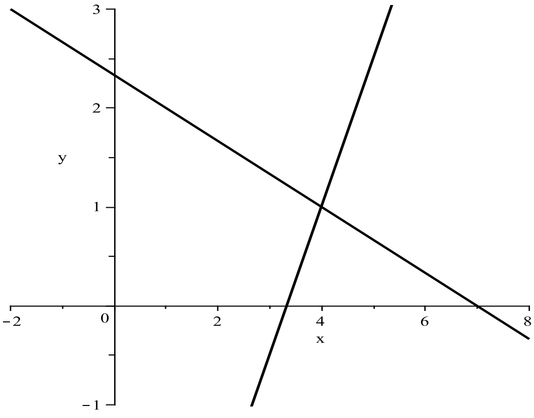
From (2), $x + 2y = 28$. Let $x = 4$ and $y = 12$, so $x + 2y = 4 + 24 = 28$ as required.

Hence the solution is correct.

Question *5.2.2* Solve the following simultaneous equations using substitution.

$$3x - 2y = 10 \quad (1)$$

$$x + 3y = 7 \quad (2)$$



Solving simultaneous equations using elimination.

- *Given two equations, add a multiple of one equation to a multiple of the other equation, thus eliminating one of the variables.*
- *Solve the resulting equation for the remaining variable.*
- *Substitute the value of that variable into either original equation, giving the value for the remaining variable.*

***Example** 5.2.3* Solve:

$$\begin{aligned} x - 3y &= -1 && (1)\\ 3x + 2y &= 8 && (2) \end{aligned}$$

Let's eliminate y. Multiply **both sides** of equation (1) by 2 and multiply **both sides** of equation (2) by 3. Then:

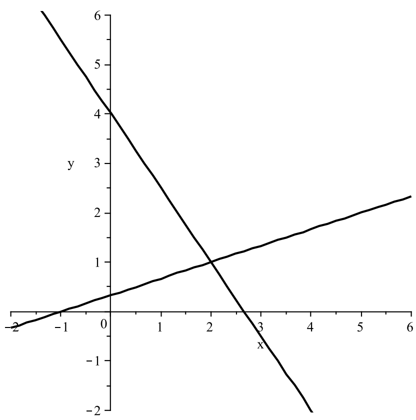
$$\begin{aligned} 2x - 6y &= -2 && (3)\\ 9x + 6y &= 24 && (4) \end{aligned}$$

Now add the left-hand sides of equations (3) and (4), and the right-hand sides of equations (3) and (4):

$$\begin{aligned} 2x - 6y + 9x + 6y &= -2 + 24\\ \Rightarrow \qquad 11x + 0y &= 22 \end{aligned}$$

Hence $11x = 22$, so $x = 2$. Substitute this back into (1) *or* (2) to get y. In (1):

$$\begin{aligned} x - 3y &= -1\\ \Rightarrow \quad 2 - 3y &= -1\\ \Rightarrow \quad 2 &= 3y - 1\\ \Rightarrow \quad 3 &= 3y\\ \Rightarrow \quad y &= 1 \end{aligned}$$

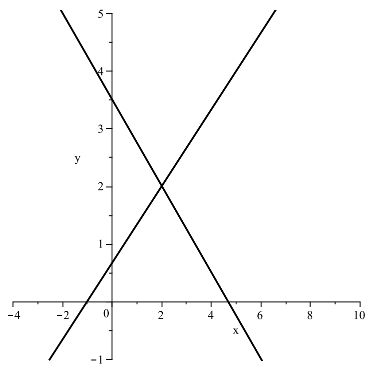


Hence the answer is $(x, y) = (2, 1)$. (Check it!)

- Take time to understand why elimination works (refer to the previous example if you like).
- When both sides of an equation are multiplied by the **same number**, the equation is effectively unchanged.
- (Recall that as long as we do the same thing to each side, the equation stays the same.)
- If we have two equations of the form $LHS_1 = RHS_1$ and $LHS_2 = RHS_2$ (where LHS means left-hand side and RHS means right-hand side), then it **must** be the case that $LHS_1 + LHS_2 = RHS_1 + RHS_2$.
- The only tricky step with elimination, is working out by what constant each equation should be multiplied.
- Remember that when you add the equations, you want one of the variables to be eliminated.
- Hence you need to choose your constants so that the coefficients of one of the variables will cancel when the equations are added.
- In the previous example, the coefficients of y were -3 and 2. When these are multiplied by 2 and 3 respectively, the new coefficients became -6 and 6, which cancel.
- You can choose any useful number(s) as your constants (except 0).
- The constants will be different for each equation.
- The constants can be positive or negative.
- The constants can be equal to 1 if you like (in which case the equation remains unchanged).

***Question** 5.2.4* Use elimination to solve the simultaneous equations:

$$3x + 4y = 14 \quad (1)$$
$$2x - 3y = -2 \quad (2)$$



- In general, you can choose either substitution or elimination to solve your equations.
- If it is easy to rewrite either equation as $x = \ldots$ or $y = \ldots$ then it will be easier to use substitution.
- If you cannot rewrite either equation like that, then elimination will be easier.

- Earlier, we saw that simultaneous equations always have either one solution, or no solutions (parallel lines), or an infinite number of solutions (identical lines).
- The following examples show what happens when there are no solutions or an infinite number of solutions.

***Example** 5.2.5* Solve the simultaneous equations:

$$
\begin{aligned}
3x - 2y &= 4 & (1)\\
-9x + 6y &= -12 & (2)
\end{aligned}
$$

Answer: Multiply (1) by 3, giving:

$$9x - 6y = 12 \qquad (3)$$

and then add (2) to (3), giving:

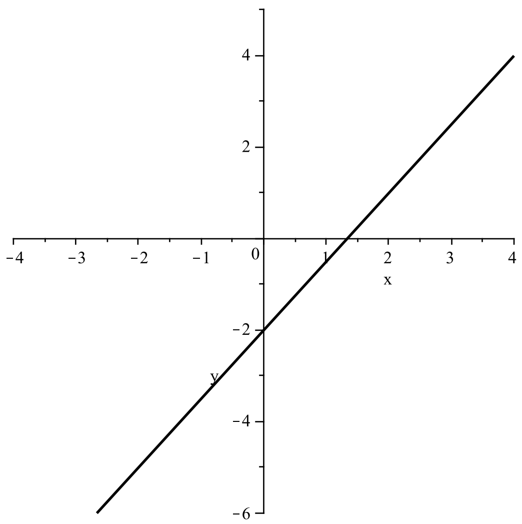
$$
\begin{aligned}
-9x + 6y + 9x - 6y &= -12 + 12,\\
\text{so :} \qquad 0 &= 0
\end{aligned}
$$

The final statement $(0 = 0)$ is **always** true.

This means that there are an infinite number of solutions to the given equations. Any point which satisfies the first equation also satisfies the second equation.

Graphically, any point on the first line is also on the second, so the lines are identical.

(On the following diagram, the lines are superimposed.)



Example *5.2.6* Solve the simultaneous equations:

$$2x - y = 3 \quad (1)$$

$$-4x + 2y = -5 \quad (2)$$

Answer: Multiply (1) by 2, giving:

$$4x - 2y = 6 \quad (3)$$

and then add (2) to (3), giving:

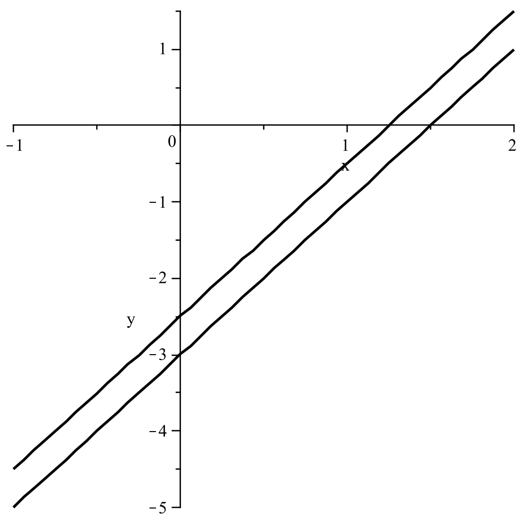
$$-4x + 2y + 4x - 2y = -5 + 6,$$

$$\text{so}: \quad 0 = 1$$

The final statement ($0 = 1$) is **never** true (that is, whatever values are chosen for x and y, the equations lead to a false statement).

This means that there is **no** solution to the given equations.

Graphically, the two lines do not intersect: they are parallel.



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6 Functions

Why are we covering this material?

- This section gives an introduction to functions and function notation.
- Most mathematics involves expressing concepts and relationships using functions.
- The notation is not too hard, but many find it quite confusing at first.
- In particular, many people find domain and range to be difficult.
- You **must** be familiar with functions and their notation: we will use it very heavily for the remainder of this semester.
- It will also be used extensively in any maths work you do in business or economics, engineering and information technology, genetics and biology.
- **Topics in this section are**
  - Functions and function notation.
  - Domain and range.
  - Composition of functions.

6.1 Functions and function notation

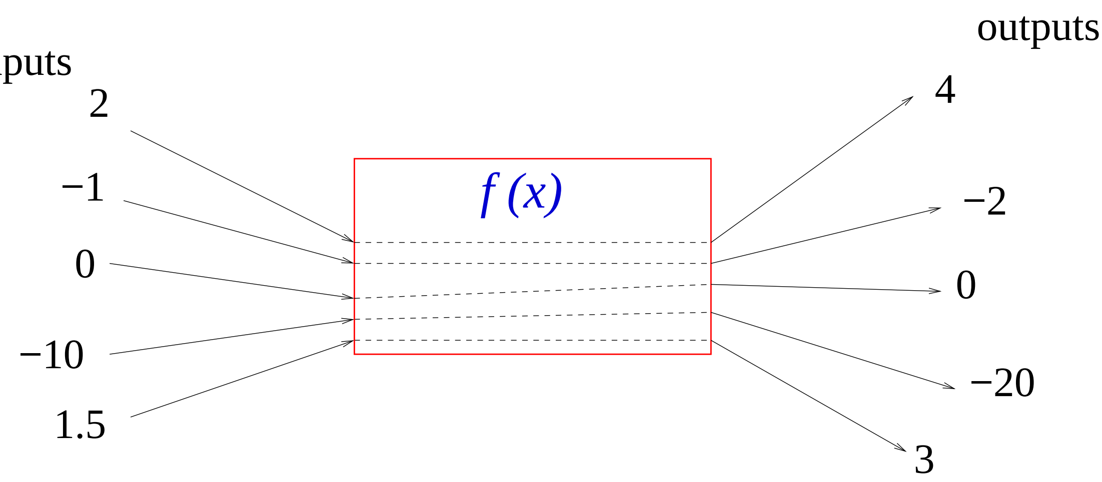
- We have seen equations like $y = 3x + 4$.
- Given a value of x, we can easily substitute the value of x into the equation to calculate y.
- There is a more formal way of writing this, using functions. A *function* specifies a **rule** by which an **input** is converted to a **unique output**.
- We have already encountered many functions (although we did not call them functions). For example:
  - the function $y = x^2$ takes the value of x as its input, and the output is equal to $x \times x$.
  - absolute value of x, written $|x|$, again takes a value x as its input, and the output is equal the distance of x from 0 (so is always positive).
- Functions are given names like f, g or h.

***Example** 6.1.1* Here are three examples of functions:

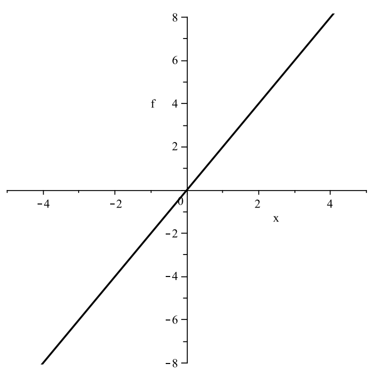
- $f(x) = x^2$ (pronounced "f of x equals x squared") is a function, called f. It takes an input x, and converts it to an output equal to x^2.
- $g(x) = |x|$ (pronounced "g of x equals the absolute value of x") is a function, called g. It takes an input x, and converts it to an output equal to $|x|$.
- $h(x) = \sqrt{x}$ (pronounced "h of x equals the square root of x") is a function, called h. It takes an input x, and converts it to an output equal to $\sqrt{x}$.

- Functions are always written like those in Example 6.1.1.
- The name of the function is given on the left, followed by a letter in brackets, followed by an equals sign, followed by an expression that (usually) involves the letter.
- The expression shows the output value to which the input is converted by the function.
- A function is sometimes likened to a 'magic box' that converts an input value to an output value by following a given rule.

***Example** 6.1.2* Let $f(x) = 2x$, so f is a function that doubles its input. Some input and output values are:



- The function $f(x)$ is represented by the box.
- f converts each input value (on the left) to a unique output value (on the right).
- For example:
  - when the input value equals -1, f gives output $2\times -1 = -2$.
  - when the input value equals 1.5, f gives output $2\times 1.5 = 3$.



***Question* 6.1.3** Define the given functions:

(1) f is a function that multiplies its input by 5.

(2) g is a function that doubles its input and adds 3.

(3) h is a function that takes the negative of its input.

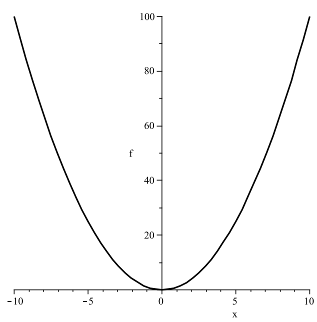
(4) f_1 is a function that leaves its input unchanged.

(5) f_2 is a function that changes its input to 4.

- Given a function $f(x) = x^2$, the thing that confuses many people is the exact meaning of the x in $f(x)$.
- The x is simply a symbol to help illustrate the definition and action of the function. It means "whatever is used as the input to f".
- If $f(x) = x^2$, then the notation means that **whatever is used as the input to** f must be **squared**.
- The **input** value is **represented** by x, but it could be **anything**.
- The **action** of the function is given by the expression x^2.

***Example* 6.1.4** Let $f(x) = x^2$. Then:

$f(3) = 3^2 = 9$	$f(0) = 0^2 = 0$
$f(-1) = (-1)^2 = 1$	$f(100) = 100^2 = 10000$
$f(0.5) = 0.5^2 = 0.25$	$f(-10) = (-10)^2 = 100$



- In the previous example we substituted numbers into f.
- Compare the next example with the previous one.

Example *6.1.5* Let $f(x) = x^2$. Then:

$f(a) = a^2$ $\qquad f(xy) = (xy)^2 = x^2y^2$

$f(3a) = (3a)^2 = 9a^2$ $\qquad f(\sqrt{a}) = (\sqrt{a})^2 = a$

$$f(x+2) = (x+2)^2 = (x+2)(x+2) = x^2 + 4x + 4$$

- Another thing to understand is that the notation $f(x) = x^2$ means **exactly the same thing** as $f(a) = a^2$ or $f(t) = t^2$.
- That is, the x is simply a convenient mathematical way of representing the input.
- (We have seem this before. If you think back to sigma notation, $\sum_{i=1}^{4} i$ is exactly the same as $\sum_{j=1}^{4} j$. The variables i and j were only used to represent the action of the sum. In functions, the x only represents the action of the function.)

Example *6.1.6* Understand the following:

- If $f(x) = 2x$ then $f(4) = 2\times 4 = 8$.
- If $f(a) = 2a$ then $f(4) = 2\times 4 = 8$.
- If $f(a) = 2a + 3$ then $f(4) = 2\times 4 + 3 = 11$.
- If $g(t) = t^2 + 1$ then $g(3) = 3^2 + 1 = 10$.
- If $f(x) = 2x$ then $f(5a) = 2\times(5a) = 10a$.
- If $f(x) = 2x + 3t$ then $f(4) = 2\times 4 + 3t = 8 + 3t$.
- If $f(t) = 2x + 3t$ then $f(4) = 2x + 3\times 4 = 2x + 12$.

***Question** 6.1.7* Let $g(x) = x^2 + 2x$.
Find each of $g(3)$, $g(-1)$ and $g(a)$.

***Question** 6.1.8* Let $f(x) = \sqrt{x}$, $h(x) = 4$ and $v(t) = 4 + 3t$.

(a) Find $f(9)$, $f(4)$ and $f(0)$.

(b) Find $h(0)$, $h(-2)$ and $h(100)$.

(c) Find $v(0), v(10)$ and $v(-a)$.

- Functions can be plotted as graphs.
- Given a function $f(x) = \ldots$, x is the independent variable and is shown on the horizontal axis.
- $f(x)$ is the dependent variable, and is shown on the vertical axis.
- Sometimes this is written as $y = f(x)$.

6.2 Domain and range

- Given a function f, it is sometimes useful to think about:
  - what are all the values that could possibly be used as a valid **input** to f?; and
  - what are all the values that could possibly arise as an **output** from f?

- Some functions can take *any* number as an input value.
- Some functions can have *any* number as a possible output value.
- Other functions have restrictions on what values can be input to the function, and/or what values can possibly occur as outputs from the function.

Domain and range.

Given a function f, the entire set of numbers which can be used as valid inputs to f is called the **domain** *of f.*

Given a function f, the entire set of numbers which can possibly arise as output values from f is called the **range** *of f.*

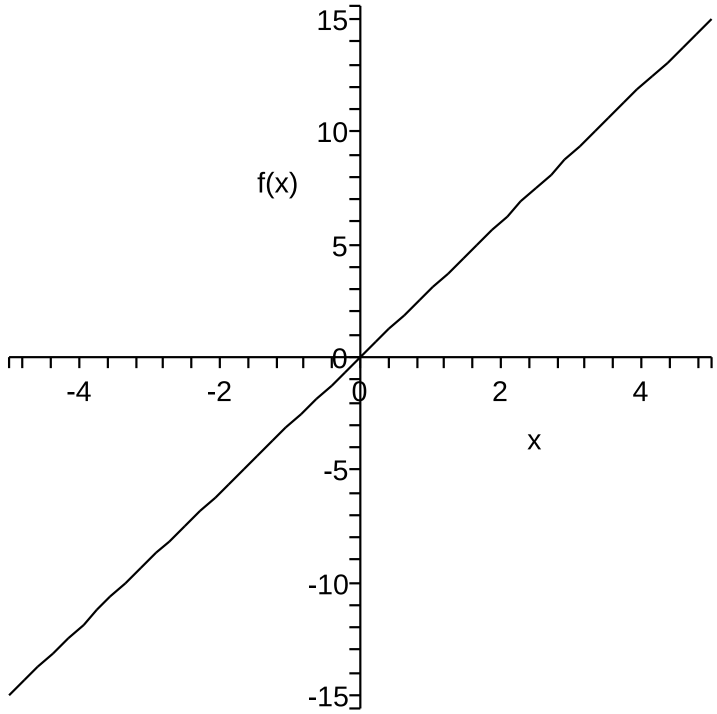
- Thus the domain is the set of all possible x values that can be used as inputs, and the range is the set of all possible y values that arise as outputs.
- The domain and range of a function are often specified in interval format. Sometimes there will be two intervals that do not overlap. In this case we use the symbol $\cup$ (set notation symbol for 'union' or 'or').
- Examples 6.2.1 and 6.2.2 and Question 6.2.3 cover domain and range. In each case a function is given, followed by
  (1) a table containing a few input values and the corresponding output value;
  (2) a graph of the function;
  (3) a diagram of the domain and range of the function; and
  (4) the domain and range of the function given in interval format.

***Example** 6.2.1* Let $f(x) = 3x$.

(1)

	-2	$\longrightarrow$	$f(x)$	$\longrightarrow$	-6	
inputs	0	$\longrightarrow$	$f(x)$	$\longrightarrow$	0	outputs
	2	$\longrightarrow$	$f(x)$	$\longrightarrow$	6	

(2)



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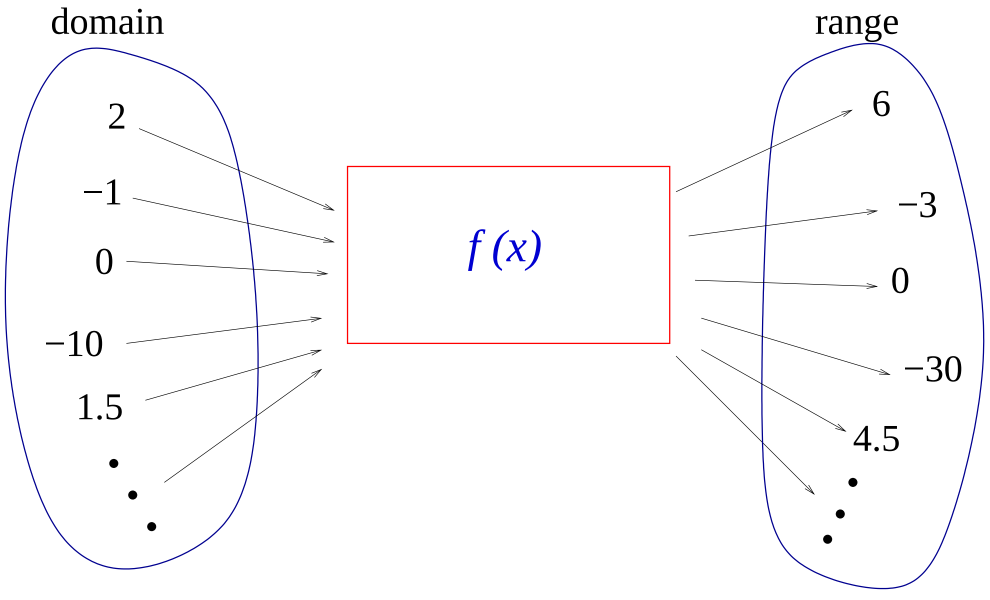


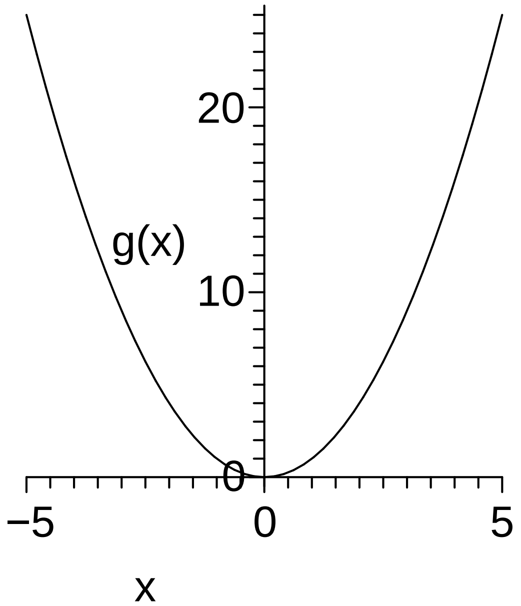
(4) The function $f(x)$ can take **any** number as its input, so its domain is $(-\infty, \infty)$. Similarly, $f(x)$ can give **any** number as its output, so its range is $(-\infty, \infty)$.

***Example** 6.2.2* Let $g(x) = x^2$.

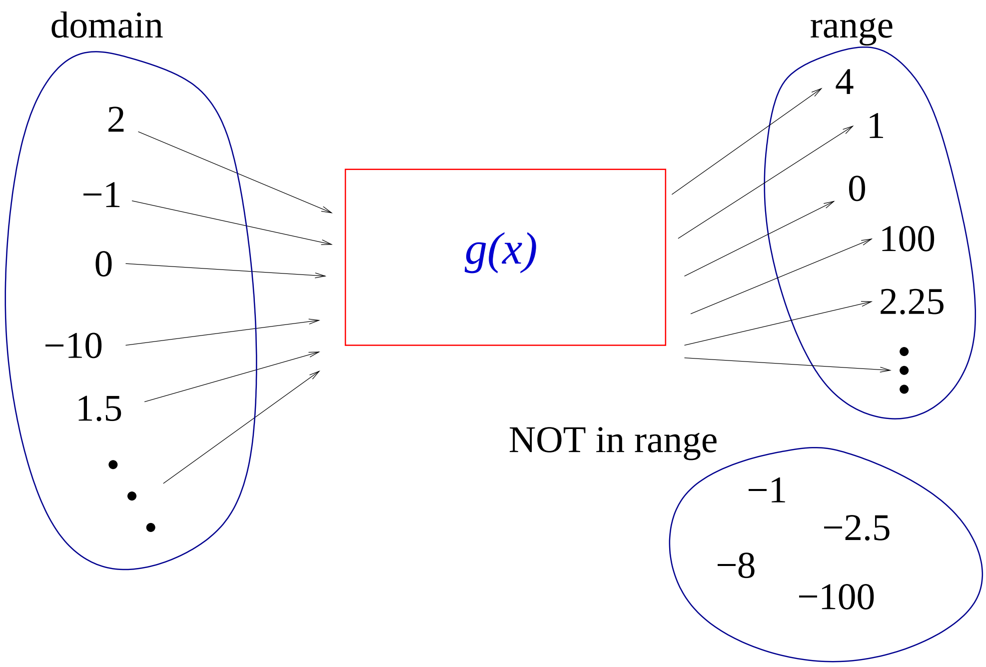
(1)

	-2	$\rightarrow$	$g(x)$	$\rightarrow$	4	
inputs	0	$\rightarrow$	$g(x)$	$\rightarrow$	0	outputs
	2	$\rightarrow$	$g(x)$	$\rightarrow$	4	

(2)



(3)



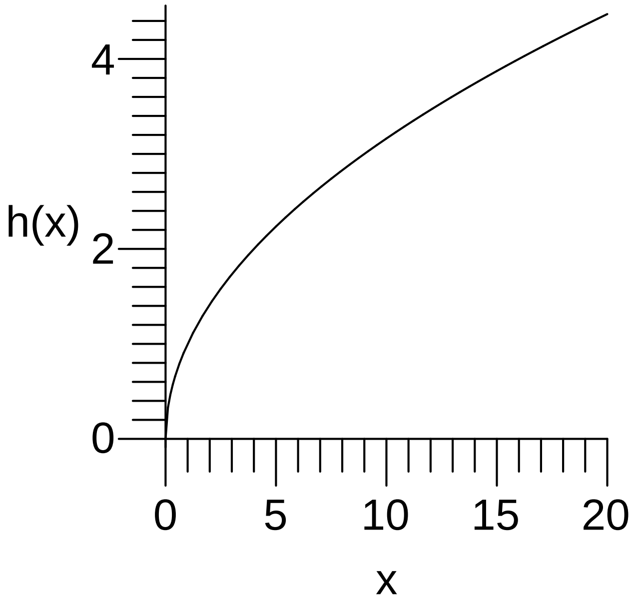
(4) The function $g(x)$ can take **any** number as its input, so its domain is $(-\infty, \infty)$. However, $g(x)$ can **never** output a negative number, so its range is $[0, \infty)$.

***Question** 6.2.3* Let $h(x) = \sqrt{x}$.

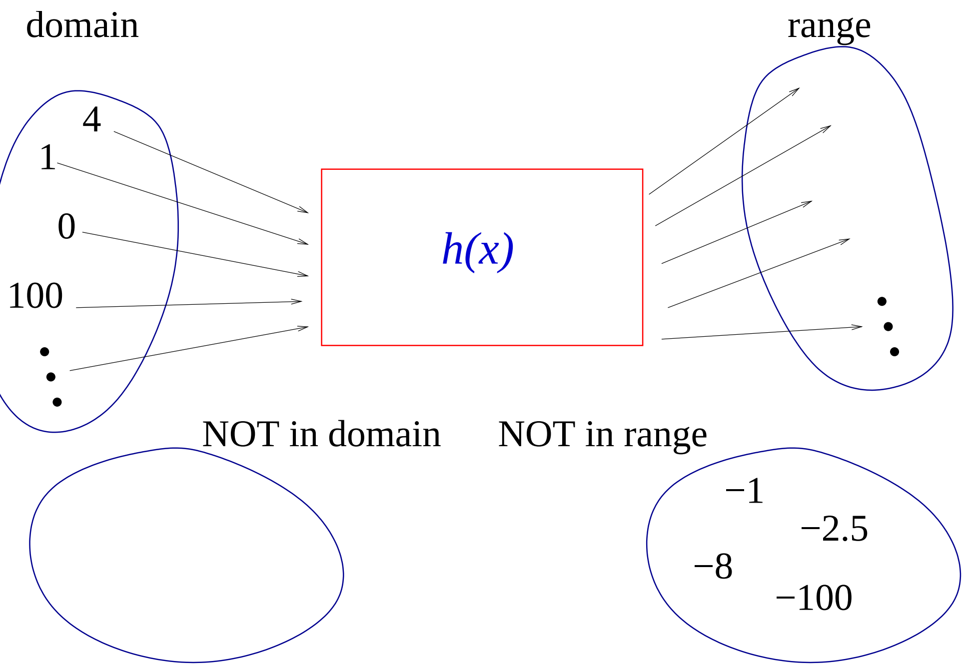
(1)

	0	$\rightarrow$	$h(x)$	$\rightarrow$	0	
inputs	4	$\rightarrow$	$h(x)$	$\rightarrow$	2	outputs
	9	$\rightarrow$	$h(x)$	$\rightarrow$	3	

(2)



(3) Fill in the missing sections: **range** and **NOT in domain**.



(4) Find the domain and range of $h(x)$.

- Many people have trouble with domain and range.
- You will be given various functions and asked to find their domain and range.
- Here are some tips for finding the **domain** of a function.
- Ask yourself: does the function have any square roots or fractions? If so then note:
  - the domain cannot contain any value which gives a negative inside a square root sign (you can't find the square root of a negative number).
  - the domain cannot contain any value which gives 0 as the denominator of a fraction (since you can't divide by 0).
- Sometimes it is easier to look at what values **are not** valid inputs, and the domain is the set of all other values.

- Here are some tips for finding the **range** of a function.
  - Is the domain restricted? If so it may restrict the range.
  - If there are any expressions involving square roots, absolute values or squaring (eg x^2) in the equation, then remember that:
    * by convention, the square root of a number is never negative: it can only be positive or 0.
    * the absolute value of any number is never negative.
    * every number squared is always 0 or positive.

***Example** 6.2.4* Let $f(x) = \sqrt{x+1}$.

We can only find the square root of a number that is ≥ 0. Hence $x + 1 \geq 0$, so $x \geq -1$. Hence the domain is $[-1, \infty)$.

Square root is always ≥ 0, so the range is $[0, \infty)$.

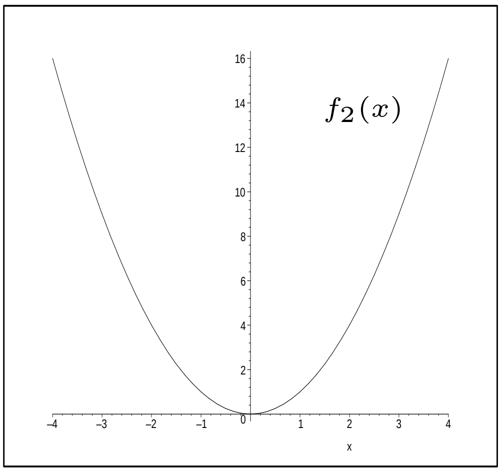
Question *6.2.5* Find the domain and range of the following:

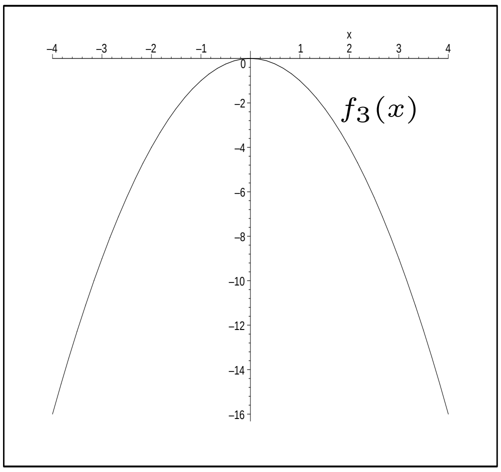
Function	domain	range
(1) $f_1(x) = x + 1$		
(2) $f_2(x) = x^2$		
(3) $f_3(x) = -x^2$		
(4) $f_4(x) = x^2 + 3$		
(5) $f_5(x) = \lvert x \rvert$		
(6) $f_6(x) = \sqrt{-x}$		
(7) $f_7(x) = \dfrac{1}{x}$		(too hard)
(8) $f_8(x) = \dfrac{1}{x^2 - 1}$		(too hard)
(9) $f_9(x) = 4$		

- It is important to understand the relationship between a function's domain/range and its graph.
- The domain is all possible values that can be used as inputs to the function, so corresponds to x values. If you draw the graph, then the graph
  - should exist at every x point in the domain; and
  - should not exist at any x point not in the domain.
- Similarly, the range is all possible values that can arise as outputs from the function, so corresponds to y values. If you draw the graph, then the graph
  - should exist at every y point in the range; and
  - should not exist at any y point not in the range.

***Example** 6.2.6* The graphs of f_2 (left) and f_3 (right) from Question 6.2.5 are shown below.

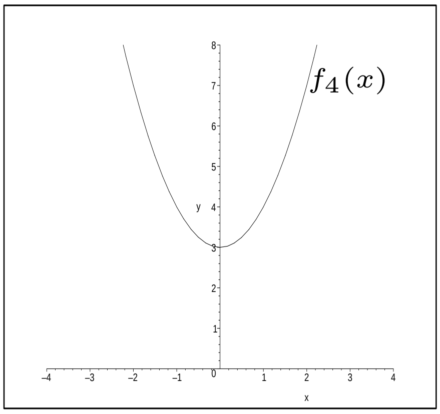
- For $f_2(x) = x^2$: the domain was $(-\infty, \infty)$, and the graph exists for every value of x; the range was $[0, \infty)$, and the graph only exists for $y \geq 0$.
- For $f_3(x) = -x^2$: the domain was $(-\infty, \infty)$, and the graph exists for every value of x; the range was $(-\infty, 0]$, and the graph only exists for $y \leq 0$.

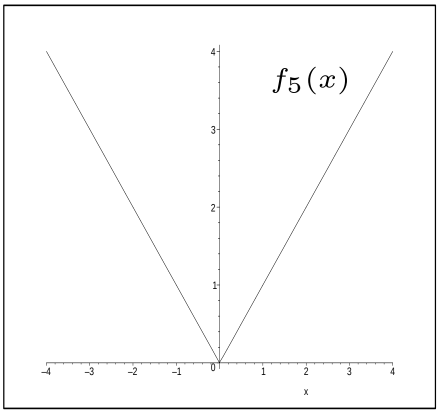




Example *6.2.7* The graphs of f_4 (left) and f_5 (right) from Question 6.2.5 are shown below.

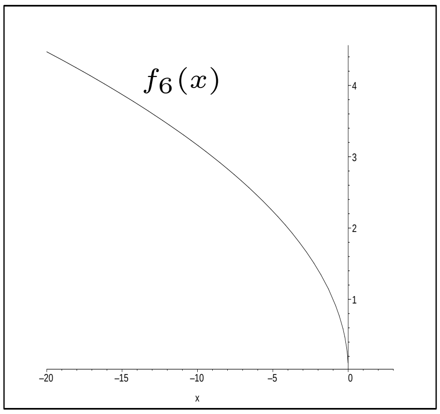
- For $f_4(x) = x^2 + 3$: the domain was $(-\infty, \infty)$, and the graph exists for every value of x; the range was $[3, \infty)$, and the graph only exists for $y \geq 3$.
- For $f_5(x) = |x|$: the domain was $(-\infty, \infty)$, and the graph exists for every value of x; the range was $[0, \infty)$, and the graph only exists for $y \geq 0$.

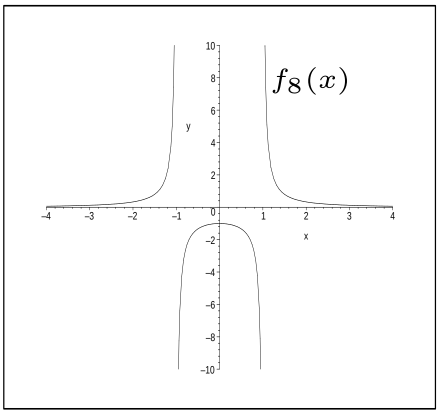




Example *6.2.8* The graphs of f_6 (left) and f_8 (right) from Question 6.2.5 are shown below.

- For $f_6(x) = \sqrt{-x}$: the domain was $(-\infty, 0]$, and the graph only exists for $x \leq 0$; the range was $[0, \infty)$, and the graph only exists for $y \geq 0$.
- For $f_8(x) = \frac{1}{x^2-1}$: the domain was $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$, and the graph exists for every value of x apart from ± 1.





6.3 Composition of functions

- Given a function $f(x)$, we can substitute numbers and letters into f. Sometimes we want to substitute other functions of x into f.

Question *6.3.1* If $f(x) = 2x$, find each of the following:

(1) $f(4)$

(2) $f(3b)$

(3) $f(3x)$

(4) $f(x^2)$

(5) $f(x+h)$

Compare the last three questions with the next three!

Question *6.3.2* Let $f(x) = 2x$, $u(x) = 3x$, $v(x) = x^2$ and $w(x) = x + h$. Find each of the following:

(1) $f(u(x))$

(2) $f(v(x))$

(3) $f(w(x))$

Composition of functions.

Given two functions $f(x)$ and $g(x)$, we can talk about:

- *the composition of f with g, $f(g(x))$, which is obtained by substituting $g(x)$ into f.*
- *the composition of g with f, $g(f(x))$, which is obtained by substituting $f(x)$ into g.*

Note that in general, $f(g(x))$ is **not** *the same as $g(f(x))$.*

- Composition of functions is sometimes called *function of a function*.

***Example** 6.3.3* Let $f(x) = 2x$ and $g(x) = x + 1$. Find
(a) $f(g(3))$ **(b)** $f(g(x))$ **(c)** $g(f(x))$

Answers:

(a) $g(3) = 3 + 1 = 4$,
hence $f(g(3)) = f(4) = 2\times 4 = 8$.

(b) $g(x) = x + 1$,
hence $f(g(x)) = f(x + 1) = 2\times(x + 1) = 2x + 2$.

(c) $f(x) = 2x$,
hence $g(f(x)) = g(2x) = 2x + 1$.

***Question** 6.3.4* Let $f(x) = 3x^2$ and $g(x) = x - 1$. Find:
(a) $f(g(0))$ **(b)** $f(g(1))$ **(c)** $f(g(x))$
(d) $g(f(0))$ **(e)** $g(f(1))$ **(f)** $g(f(x))$

NOTES

NOTES

7 Quadratic equations and polynomials

Why are we covering this material?

- We have seen linear equations, with a single solution.
- Now we look at equations with higher powers of x, and (possibly) multiple solutions.
- We'll see how to solve quadratics (using a formula or factorisation), and the shape of the graphs of some polynomials.
- Polynomials are used extensively in real-world modelling, particularly by engineers, biologists and economists.
- We'll use them for much of the rest of this course. Be familiar with the notation, and in particular be able to solve quadratic equations.
- **Topics in this section are**
  - Introduction to polynomials.
  - Quadratics.
  - Shapes of some polynomial functions.
  - Solving quadratics using the Quadratic Formula.
  - Solving quadratics by factoring.
  - Applications of quadratics.

7.1 Introduction to polynomials

- We've seen equations of straight lines: $y = mx + c$.
- Using function notation, we can write this equation as $f(x) = mx + c$.
- In any linear equation, the highest power of x is 1.
- What if our function involves terms with x raised to integer powers larger than 1?
- For example, $P(x) = x^2 - 3x + 2$ includes an x^2 term.

Polynomials

Any function that involves a variable raised only to powers that are positive integers, is called a **polynomial**.

The **degree** *of a polynomial $P(x)$ is the highest power of x in $P(x)$.*

A polynomial of degree 1 is called **linear**, *a polynomial of degree 2 is called a* **quadratic** *and a polynomial of degree 3 is called a* **cubic**.

Example *7.1.1* Examples of polynomials include:

- Linear: $2x - 1$, $-3x + 7$
- Quadratic: $x^2 + 2x + 1$, $2x^2 + 1$, $(x - 2)(x + 1)$
- Cubic: $x^3 + 2x^2 + 2x + 1$, $x^3 - 4$
- Degree 4: $7x^4 - 2x^3 + x^2 - 8x + 4$
- Degree 5: $-6x^5 - 7x$

- All polynomials can be written as functions. For example, if $y = x^2 + 2x + 1$, then we could write $f(x) = x^2 + 2x + 1$.
- All polynomials involve a variable, usually x. However, the variable can be t, or anything else.
- Given a value for x, you can evaluate the polynomial at that x value simply by substituting x into the function.
- Some terminology is important:

***Example** 7.1.2* In the polynomial $5x^3 - 4x^2 + 3$,
the "x^2 term" is $-4x^2$,
the coefficient of x^3 is 5,
the coefficient of x is 0,
and the constant term is 3.

- We have already covered linear equations in detail. Here we concentrate on quadratics.

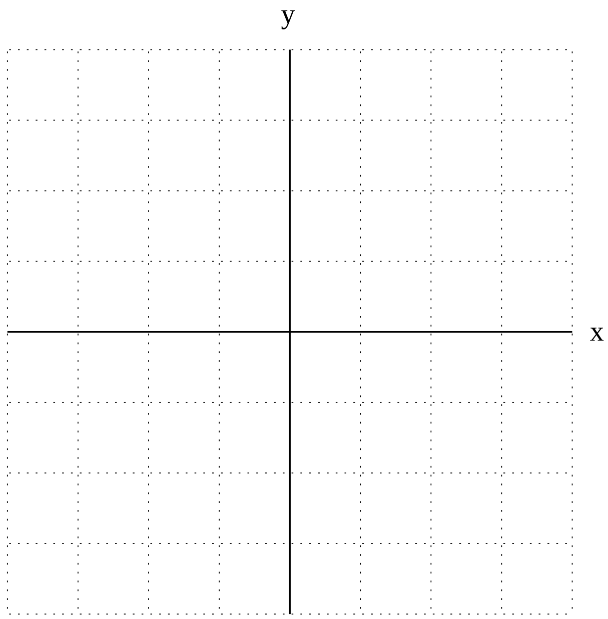
7.2 Quadratics

- Quadratics are polynomials in which the **highest power of the variable is 2**.
- The general form of a quadratic is: $f(x) = ax^2 + bx + c$, where a, b and c are constants and x is the variable.
- You need to be able to sketch graphs of quadratics.
- This can be done in the same way as for straight lines: calculate some points on the graph, mark them on some axes, and 'join the dots'.
- Make sure you find enough points to be clear how the graph looks. For quadratics, two points will not be enough.

Question *7.2.1* Sketch $p(x) = x^2 - 3x + 2$.

(Note: we could rewrite $p(x)$ as $p(x) = (x - 1)(x - 2)$.)

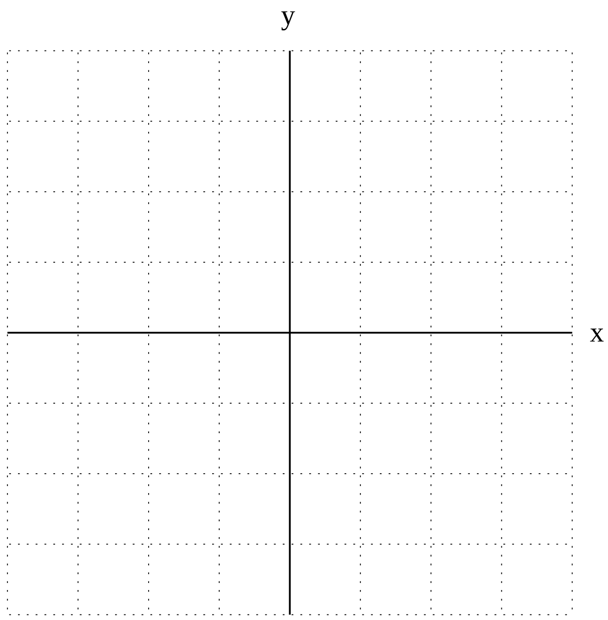
x	-1	0	1	2	3	4
$p(x)$	6	2	0	0	2	6



Question *7.2.2* Sketch the graph of $p(x) = x^2 + 1$.

(Note: y is never zero or negative, for *any* x value.)

x	-3	-2	-1	0	1	2	3
$p(x)$	10	5	2	1	2	5	10



- Compare the two graphs. The first graph crosses the x-axis in two places; the second does not cross at all.
- The places where the graph of a polynomial crosses the x-axis are important.

Roots (solutions) of an equation.

The **roots** *or* **solutions** *of an equation are those values of x for which $y = 0$; thus they are the points at which the graph crosses the x axis.*

Solving *an equation involves finding all its roots.*

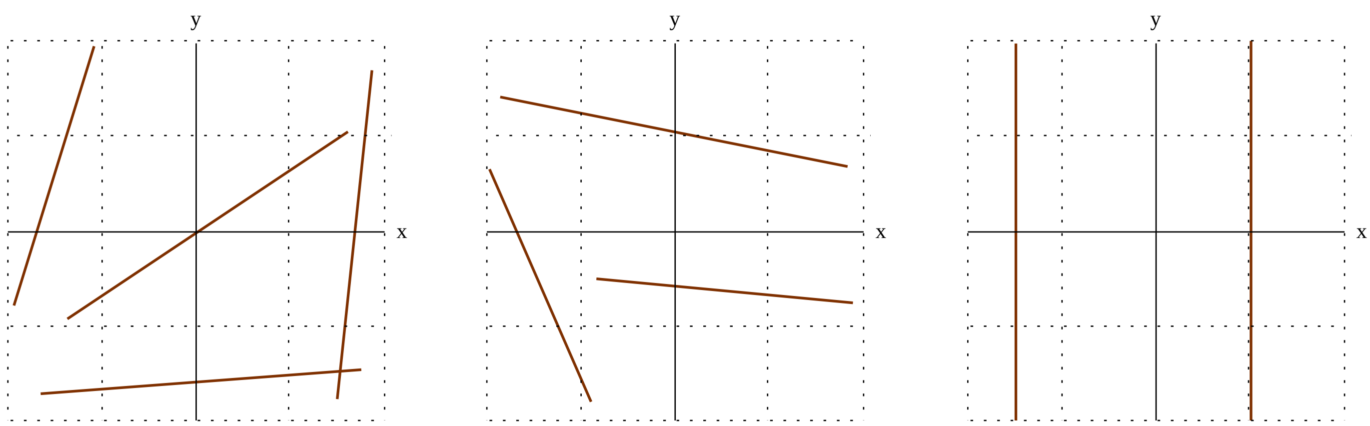
- Shortly we'll see how to solve quadratic equations.

7.3 Shapes of some polynomial functions

- You need to be familiar with the general shapes of the graphs of linear and quadratic polynomials.

***Example** 7.3.1* Consider a **linear** equation $y = ax + b$. (note: previously we have written this as $y = mx + c$; this means the same thing, we just give different names to the constants.)

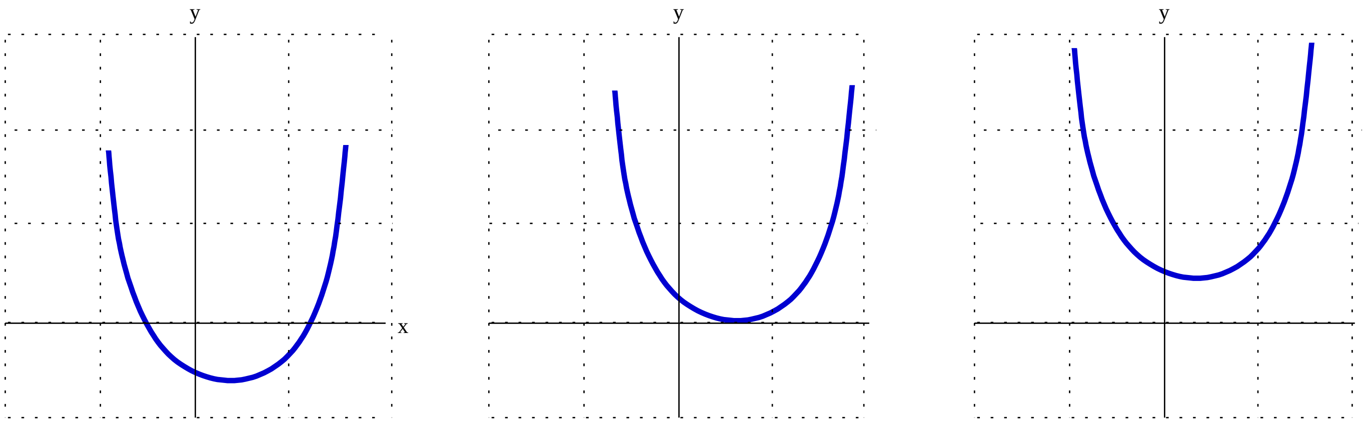
When the gradient a is positive, the line goes up to the right (see the left-hand axes). When the gradient a is negative, the line goes down to the right (see the middle axes). Lines with equation $x = c$ are vertical (see the right-hand axes).



All quadratic equations have the same general shape, called a *parabola*. Parabolas are symmetrical.

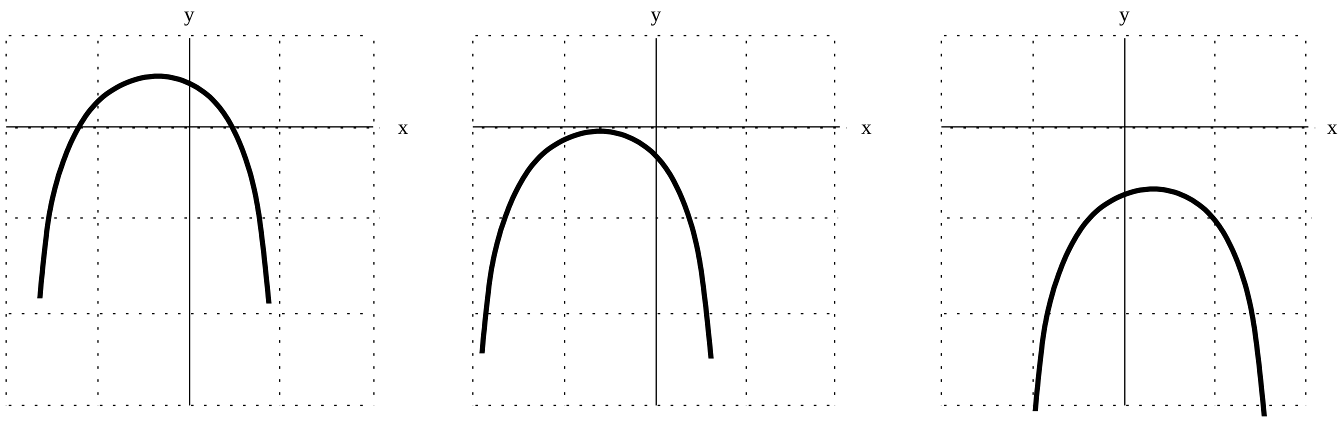
***Example** 7.3.2* Consider a **quadratic** $y = ax^2 + bx + c$, in which $a > 0$. (We might have $b = 0$ or $c = 0$.)

If a is positive in $y = ax^2 + bx + c$ then the graph looks like a 'valley'. On the left there are two roots, in the middle there is one root (or 2 equal roots), and on the right there are no roots.



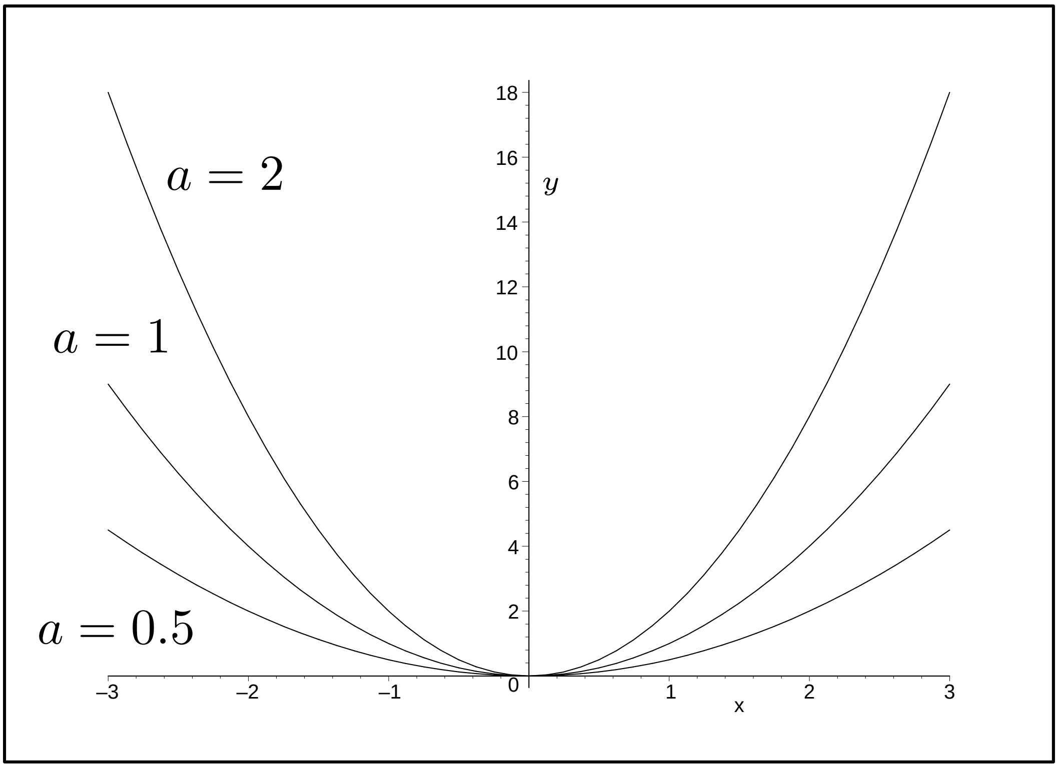
***Example** 7.3.3* Consider a quadratic $y = ax^2 + bx + c$, in which $a < 0$. (We might have $b = 0$ or $c = 0$.)

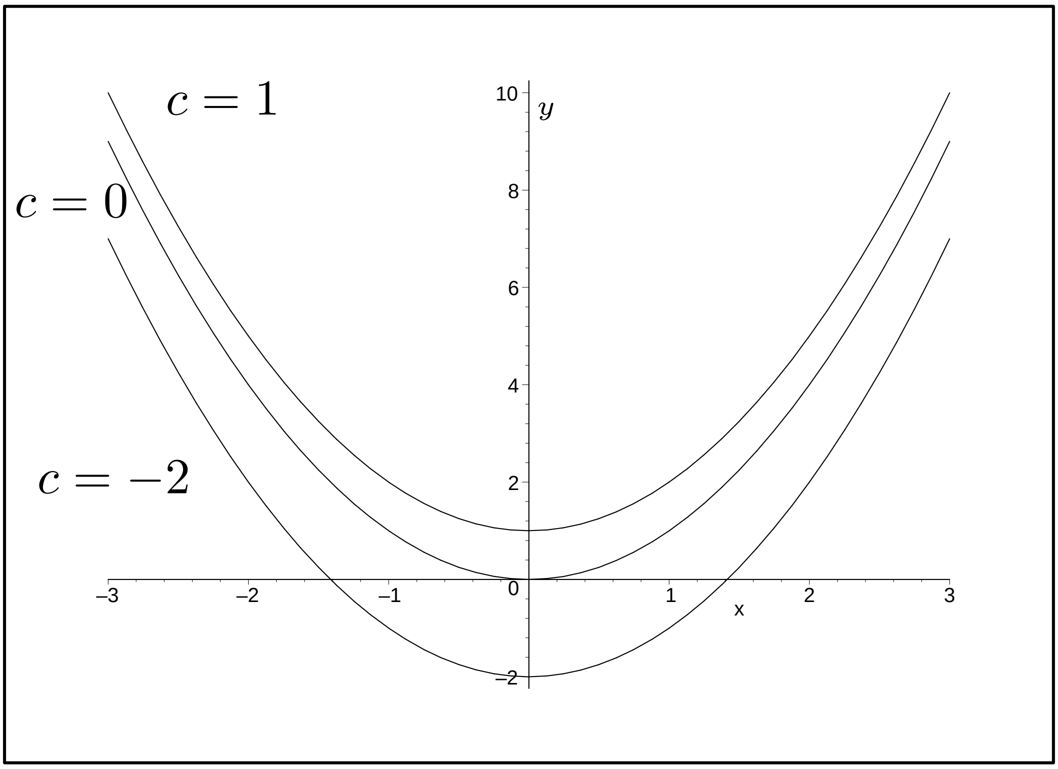
If a is negative in $y = ax^2 + bx + c$ then the graph looks like a 'hill'. On the left there are two roots, in the middle there is one root (or 2 equal roots), and on the right there are no roots.



In the equation $y = ax^2 + bx + c$,

- the size of a shows how "opened out" the curve is. (See the top diagram; larger values of a are steeper.)
- c is the y-intercept, just as for linear functions. (See the bottom diagram.)





7.4 Solving quadratics using the Quadratic formula

- Recall that a solution or root of a polynomial is a value of x which gives $y = 0$, and that solving an equation involves finding all of its roots.
- It is easy to solve a linear equation $y = mx + c$: just substitute $y = 0$ into the equation, and solve directly for x.
- Solving a quadratic $y = ax^2 + bx + c$ is harder:
  - There may be 0, 1 or 2 roots.
  - Where do we start? We can let $y = 0$, but what next?
- There is an important formula for solving quadratics.

Quadratic formula.

The roots or solutions of the quadratic $ax^2 + bx + c = 0$ *are given by:*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- We have seen that a quadratic has 0, 1 or 2 roots. This formula gives 0, 1 or 2 roots in the following way.
  - if $b^2 - 4ac > 0$, then there are **two different** values in the part of the formula $\pm\sqrt{b^2 - 4ac}$. This results in two distinct roots.
  - if $b^2 - 4ac = 0$, then there is one value in the part of the formula $\pm\sqrt{b^2 - 4ac}$ (the value is 0). This yields a single root.
  - if $b^2 - 4ac < 0$, then we are trying to take the square root of a negative number. This is not possible, so there are no roots.

- The quadratic formula is easy to use: simply substitute the values of a, b and c into the formula to get the roots. You don't need to commit it to memory: it will be given on your exam.

- To apply the formula, you **must** first write your equation in the form $\quad ax^2 + bx + c = 0.$

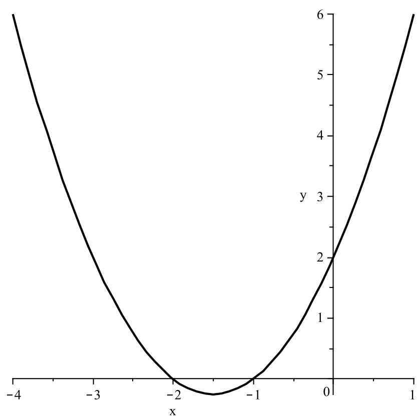
- Then a is the coefficient of x^2, b is the coefficient of x and c is the constant term.

- Be careful with negative coefficients. For example, if $-2x^2 - 3x - 4 = 0$ then $a = -2$, $b = -3$ and $c = -4$.

***Example** 7.4.1* Solve $\quad x^2 + 3x + 2 = 0.$

We have $a = 1$, $b = 3$ and $c = 2$. So:

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4\times1\times2}}{2\times1} \\ &= \frac{-3 \pm \sqrt{9-8}}{2} \\ &= \frac{-3 \pm \sqrt{1}}{2} \\ &= \tfrac{-3+1}{2} \text{ or } \tfrac{-3-1}{2} \\ &= \tfrac{-2}{2} \text{ or } \tfrac{-4}{2} \\ &= -1 \text{ or } -2 \end{aligned}$$



Hence there are two roots, $x = -1$ and $x = -2$. To check these roots, substitute each into the original equation to verify that the answer is 0.

Question *7.4.2* Solve and sketch each of the following:

(a) $2x^2 - 7x - 4 = 0$

(b) $x^2 - 4x + 4 = 0$

Question *7.4.3* Solve each of the following quadratics:

(a) $2x^2 + 3x + 3 = 0$

(b) $x^2 - 9 = 0$

(c) $x^2 = 4x$

7.5 Solving quadratics by factoring

- We have seen how the quadratic formula solves quadratics.
- Sometimes it is possible to find the solution(s) using *factoring*.
- First, you need to remember how to expand.

Example *7.5.1* Expand each of the following:

(a) $(x-2)(x-1)$

Answer:

$(x-2)(x-1) = x^2 - 1\times x - 2\times x + 2 = x^2 - 3x + 2$

(b) $(x-2)^2$

Answer: $(x-2)^2 = (x-2)(x-2) = x^2 - 4x + 4$

- From Example 7.5.1, we can see that the equation $(x-2)(x-1)$ is **exactly the same** as the equation $x^2 - 3x + 2$.
- Thus, if we are asked to solve $x^2 - 3x + 2 = 0$, it is the same as solving $(x-2)(x-1) = 0$.
- Similarly, because $(x-2)^2$ is the same as $x^2 - 4x + 4$, then solving $x^2 - 4x + 4 = 0$ is the same as solving $(x-2)^2 = 0$.
- We need to use the quadratic formula to solve the equations written without the brackets.
- However, we can easily solve equations like $(x-2)(x-1) = 0$ and $(x-2)^2 = 0$ in our heads.
- To do so, we make use of the fact that **if two things multiply to give zero, then (at least) one of the things must equal zero.**

***Example** 7.5.2* Solve $(x-2)(x-1)=0$.

Because $(x-2)\times(x-1)=0$, we must have either $(x-2)=0$ or $(x-1)=0$.

If $(x-2)=0$ then $x=2$. And if $(x-1)=0$ then $x=1$.

Hence the solutions are $x=2$ and $x=1$.

(Note that $(x-2)(x-1)=x^2-3x+2$. Hence you have shown that the solutions to $x^2-3x+2=0$ are $x=2$ and $x=1$. You would get the same solutions using the Quadratic formula.)

***Example** 7.5.3* Solve $(x-2)^2=0$.

Because $(x-2)\times(x-2)=0$, we must have either $(x-2)=0$ or $(x-2)=0$ (which is the same thing).

If $(x-2)=0$ then $x=2$.

Hence there is only one solution, $x=2$.

(Note that $(x-2)^2=x^2-4x+4$. Hence the answer you got here should match those from Part (b) of Question 7.4.2.)

- This technique for solving quadratic equations is called **factoring**.
- Recall that given a number n, factors of n are numbers which multiply together to give n. For example, $15=5\times3$.
- Here, we are using a similar technique, but rather than finding factors of a number, we are instead finding factors of the quadratic equation.
- That is, we are trying to rewrite the equation as a pair of things which multiply together to give the equation. For example, $x^2-3x+2=(x-2)\times(x-1)$.

- Given a quadratic equation, if it is already written in factored form, then don't use the quadratic formula to solve it: you can solve it just by looking at it.
- If the equation is not written in factored form, then you can either use the quadratic formula, or you can try to factor it.
- Factoring requires some skill and patience. Study the following example.

***Example** 7.5.4* Use factoring to solve $x^2 + 6x + 8 = 0$.

To factor this expression, we must rewrite it as:

$$x^2 + 6x + 8 = (x + a)(x + b),$$

where we need to find the numbers a and b.

Now $(x + a)(x + b) = x^2 + bx + ax + ab = x^2 + (a + b)x + ab$. Hence:

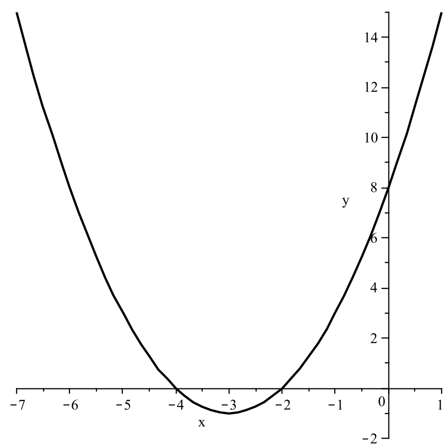
$$x^2 + 6x + 8 = x^2 + (a + b)x + ab,$$

so we must have $(a + b) = 6$ and $ab = 8$.

Then we use trial and error to find values for a and b that satisfy these equations.

First try $a = 1$ and $b = 8$. Then $ab = 8$, but $a + b = 9$, not 6. Next try $a = 2$ and $b = 4$. Then $ab = 8$, and $a + b = 6$.

Hence $x^2 + 6x + 8 = (x + 2)(x + 4)$, and when we solve this we get $x = -2$ or $x = -4$.



- Factoring is often not easy: if you are not comfortable with it, then note that the quadratic formula always works, and you are always welcome to use it.

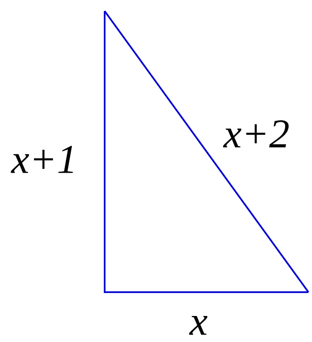
***Question** 7.5.5* Use factoring to solve $y = x^2 + x - 2$, then roughly sketch its graph.

7.6 Applications of quadratics.

- Quadratic equations can be used to solve a variety of problems.
- Be careful: when solving some problems, one of the roots may be mathematically correct, but not possible for practical reasons.
- For example, lengths or areas can't be negative.

***Example** 7.6.1* Find all right-angle triangles which have hypotenuse 2 units longer than one side and 1 unit longer than the other side.

Let the length of the shortest side be x. Then the hypotenuse is of length $x+2$, and the other side is of length $x+1$.



From Pythagoras' theorem:

$$\begin{aligned} c^2 &= a^2+b^2, \\ \text{so :}\quad (x+2)^2 &= x^2+(x+1)^2, \\ \text{so :}\quad x^2+4x+4 &= x^2+x^2+2x+1, \\ \text{so :}\quad 0 &= x^2-2x-3, \\ \text{so :}\quad 0 &= (x+1)(x-3). \end{aligned}$$

Hence $x=-1$ or 3. Clearly, $x=-1$ isn't possible for the length of a side, so the only solution is $x=3$.

Hence the other sides are of lengths $x+1=4$ and $x+2=5$, so the triangle has sides of lengths 3, 4 and 5.

Question *7.6.2* Find the dimensions of all rectangles in which the area equals the perimeter plus 3.5, and in which the longer sides are twice the length of the shorter sides.

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8 Logarithms and exponentials

Why are we covering this material?

- We've just seen polynomial functions, including quadratics and linear equations, which all involved x raised to various powers.
- Now, we encounter a different type of function: exponentials, which involve the variable x in the power.
- Exponential functions are used for:
  - modelling population increases and decreases;
  - economic growth and interest return on money;
  - radioactive decay.
- We'll see how to distinguish exponential growth functions from exponential decay functions.
- We'll introduce e^x, a special exponential function.
- Finally, we'll see how logarithms relate to exponentials.
- **Topics in this section are**
  - Introduction to exponentials.
  - Exponential growth.
  - Compound interest.
  - Exponential decay.
  - The exponential function, e^x.
  - Logarithms.

8.1 Introduction to exponentials

- Recall, from much earlier on, the *power* (or *index*) laws:

Rule	*Example*
$a^m a^n = a^{m+n}$	$2^3 2^2 = 2^5$
$(a^m)^n = a^{mn}$	$(2^3)^2 = 2^6$
$\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{2^3}{2^2} = 2^{3-2} = 2^1 = 2$
$a^{-m} = \dfrac{1}{a^m}$	$2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$
$a^0 = 1$	$2^0 = 1$
$a^{1/m} = \sqrt[m]{a}$	$2^{1/3} = \sqrt[3]{2}$

- If $y = x^n$, then x is called the *base*, and n is called the *power* or *exponent*.
- Whenever we have worked with powers so far, the **power** has been a constant, and the **base** has been the variable: for example, x^2, x^7 and x^{-1}.
- Now we introduce a new type of function, called an *exponential* function.
- Such functions involve a variable in the exponent, such as

$$f(x) = 2^x, \quad g(x) = 10^x, \quad h(x) = e^x.$$

- Polynomial functions have x in the base and constants in the power (eg $x^2 + 2x + 1$), whereas exponentials have x in the power and a constant base.

- Exponential functions have many practical uses, such as:
  - radiocarbon dating of fossils;
  - decay of nuclear waste and contamination;
  - compound interest in bank accounts;
  - population growth and decay.

8.2 Exponential growth

- Consider an exponential function $y = a^x$, where a is a constant.
- We can use almost any (positive) number a as our base. For the moment, we restrict ourselves to $a > 1$.

***Question** 8.2.1* Let $f(x) = 2^x$. Create a table of values for f for $x \in \{0, 1, 2, 3, 4\}$.

x	0	1	2	3	4
$f(x)$					

***Question** 8.2.2* At time of fertilisation, $t = 0$, a zygote contains 1 cell. This cell splits into two new cells after a certain time, and cells continue to split into two. Let the number of cells after t time periods be $C(t)$. Find $C(2)$, $C(3)$ and $C(4)$.

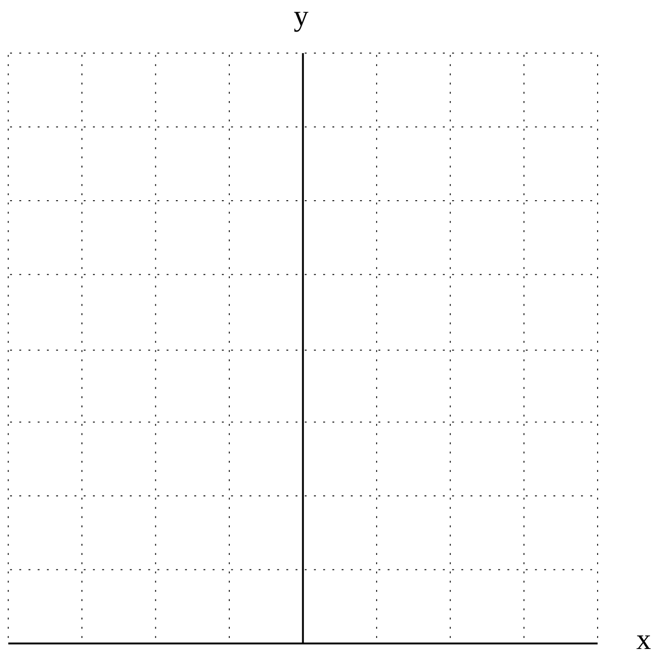
Note that these numbers match those in Question 8.2.1. Hence for any integer time $t \geq 0$, $C(t) = 2^t$.

In the previous examples x was always positive, but this does not need to be the case.

***Question** 8.2.3* Let $f(x) = 2^x$. Create a table of values for f for $x \in \{-4, -3, -2, -1\}$.

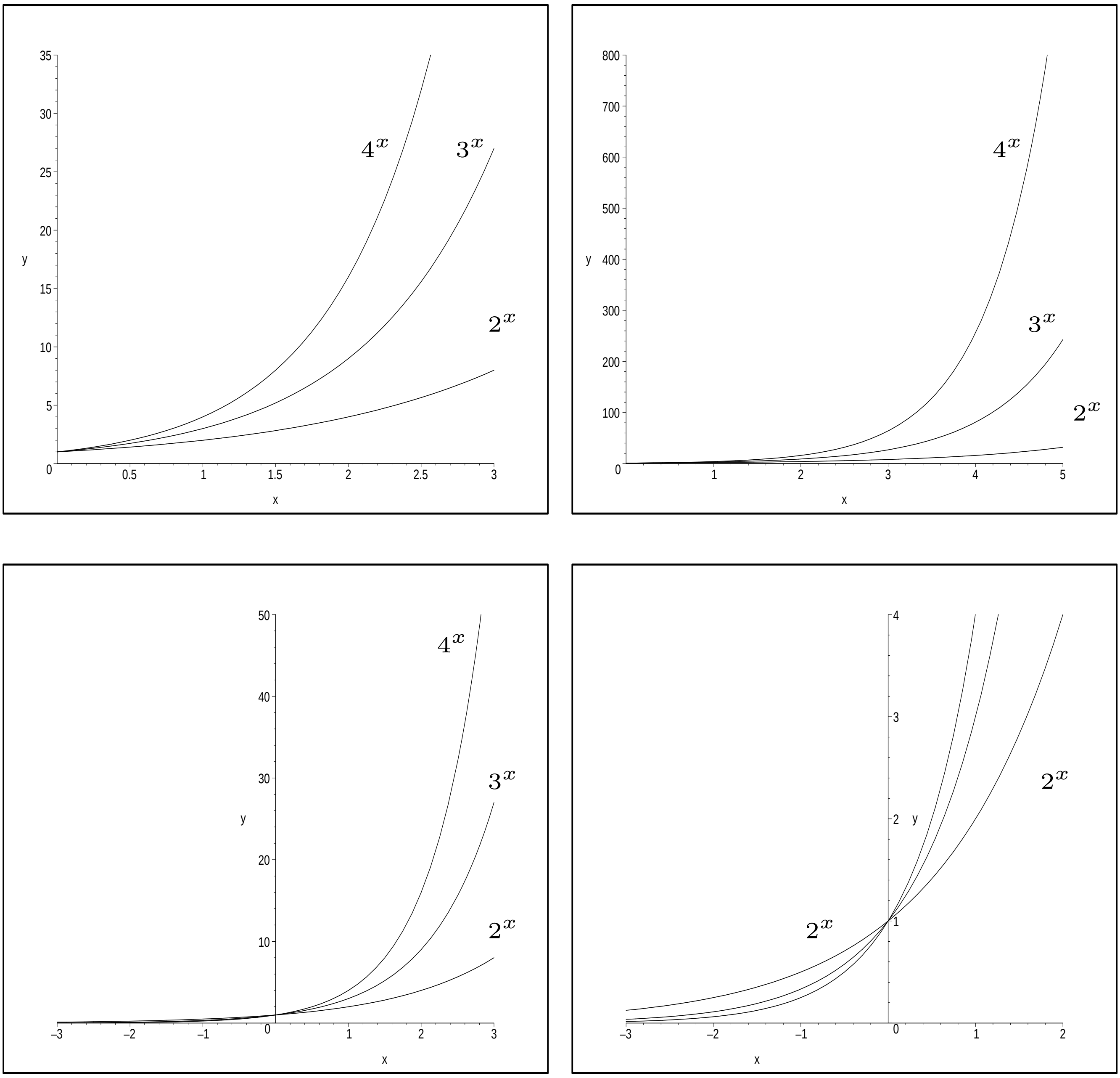
x	-4	-3	-2	-1
$f(x)$				

***Question** 8.2.4* Let $f(x) = 2^x$. Use Questions 8.2.1 and 8.2.3 to sketch a graph of $f(x)$.



- In Example 8.2.5 we plot the exponential functions 2^x, 3^x and 4^x on each of four sets of axes, for different x values.
- Observe the similarities and differences between the graphs, and compare them to the graph in Question 8.2.4.

Example *8.2.5* Plots for $x \in [0, 3]$ (at the top left), $x \in [0, 5]$ (top right), $x \in [-3, 3]$ (bottom left) and $x \in [-3, 2]$.



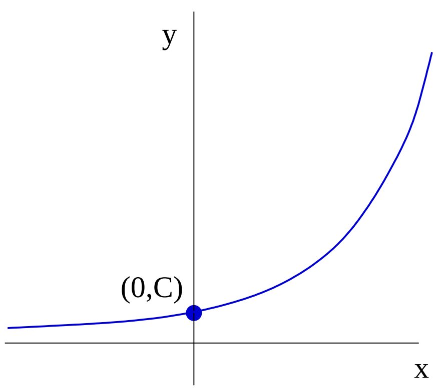
- When x is a large, negative number y gets very close to 0 but never equals 0.
- When x is large and positive, the graphs get very large and positive.
- The graphs all grow very quickly.
- The graphs all pass through the point $(0, 1)$.

- The graphs in Example 8.2.5 all include an exponential term, and y gets bigger as x gets bigger.
- Functions like this display **exponential growth**.
- Each of the graphs we have seen so far pass through the point $(0, 1)$ (because anything to the power 0 equals 1).
- Many quantities in business and nature grow exponentially.
- To accurately model such quantities, we need a more general form of the equation for exponential growth.

Exponential growth functions.

If $a > 1$ and $C > 0$ then any function of the form $y = Ca^x$ will display exponential growth.

- In the general form of the exponential growth function:
  - C is a constant representing some *initial conditions*, such as:
    - the population at time $t = 0$; or
    - the amount of radioactive material at time $t = 0$; or
    - the temperature at time $t = 0$.
  - Hence the graph passes through the point $(0, C)$.
  - The base a is a constant depending on the problem; it determines how quickly the quantity grows.
  - The graph will look like:



Example *8.2.6* There are four cockroaches in a student's house at time $x = 0$. The population size P triples every week. **(1)** Write an expression for the population size $P(x)$ after x weeks.

Answer: At time 0 there are 4 cockroaches, so we have $C = 4$. The population size triples every week, so $a = 3$.

Hence the equation is $P(x) = 4{\times}3^x$. (Remember BEDMAS: $4{\times}3^x \neq 12^x$)

(2) When is P first larger than 200?

Answer: We can use trial and error to find when the population is first larger than 200. By substituting, $P(1) = 4{\times}3 = 12$, $P(2) = 4{\times}3^2 = 36$, $P(3) = 4{\times}3^3 = 108$ and $P(4) = 4{\times}3^4 = 324$.

Hence the population is first larger than 200 after 4 weeks.

8.3 Compound interest

- Another quantity that grows exponentially is money earning compound interest in a bank account.
- The growth depends on two things: the initial amount invested and the interest rate.

Compound interest.

If P dollars are invested at an interest rate of r per time period (where r is a decimal) for a total of x time periods, then the final value $F(x)$ of the investment is given by:

$$F(x) = P\,(1 + r)^x.$$

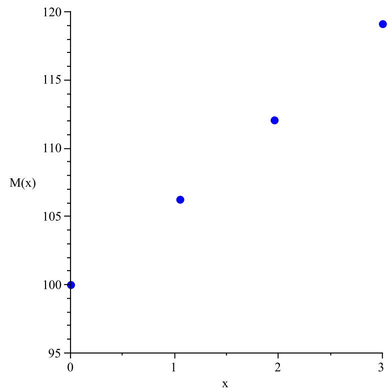
- To use the compound interest formula you may need to convert r from a percentage to a decimal number.

- Be careful with time units: a key concept is how often the interest *compounds*, which means how often the interest is added to the account.

- Usually you will be given an interest rate *per annum*. If the time periods for compounding are different, then you will need to carefully and consistently calculate r and x.

***Example** 8.3.1* $100 is invested in a bank term deposit, earning 6% interest per year, compounding annually.
(1) Write an expression for the balance M after x years.

Answer: Here we have $r = 0.06$ and $P = 100$.
Hence $M(x) = 100(1 + 0.06)^x$
$= 100(1.06)^x$.

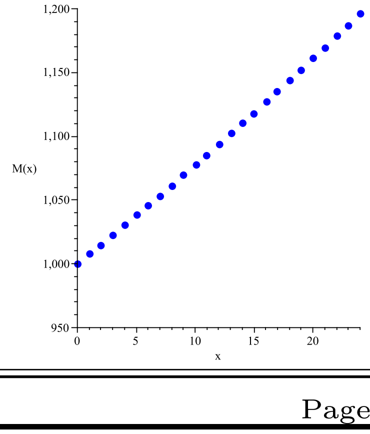


(2) What is the balance after 3 years?
Answer:
When $x = 3$, $M(3) = 100(1.06)^3$
$= \$119.10$.

***Example** 8.3.2* $1000 is invested at 9% per annum compounding monthly for 2 years. What is the final value F?

Answer: Here the interest compounds monthly, so we need to convert both x and r to monthly values. There are 12 months per year, so you need to multiply the number of years by 12, and divide the annual interest rate by 12.

Hence $x = 2\times12 = 24$ (which is the number of months in 2 years) and $r = 0.09/12 = 0.0075$ (which is the percent interest per month).



Hence $F = 1000(1 + 0.0075)^{24}$
$= \$1196.41$

Question *8.3.3* \$100 is invested in a bank term deposit, earning 6% interest per year, compounding monthly.

(a) Write an expression for the amount of money in the account after x years.

(b) What is the balance after 3 years? (use $(1.005)^{36} = 1.197$)

(c) Compare your answer to Part (2) with that to Example 8.3.1 and explain any differences.

(d) If the time for compounding keeps getting shorter, what will happen to the amount of interest earned?

8.4 Exponential decay

- Until now we have only considered exponential functions of the form $y = Ca^x$, where $C > 0$ and $a > 1$.
- All such functions represented exponential growth.
- Exponential functions can also have **negative** powers.

Question *8.4.1* Let $f(x) = 2^{-x}$ (This also equals $(1/2)^x$.)

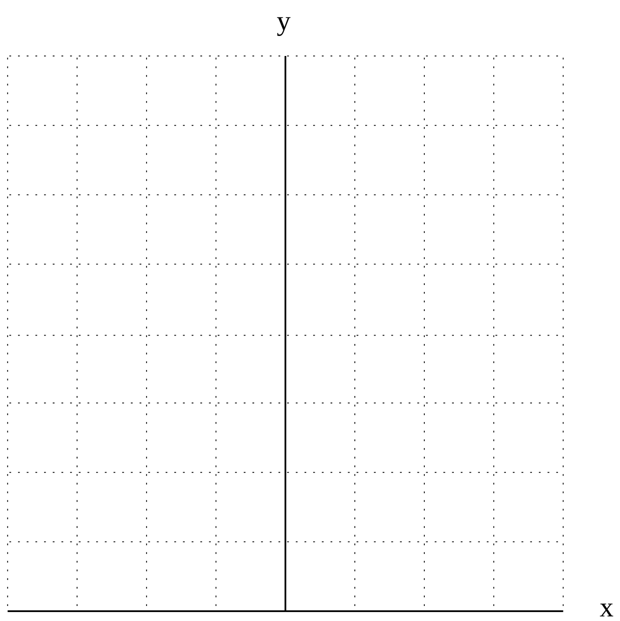
(a) Create a table of values for f for $x \in \{0, 1, 2, 3, 4\}$.

x	0	1	2	3	4
$f(x)$					

(b) Create a table of values for f for $x \in \{-4, -3, -2, -1\}$.

x	-4	-3	-2	-1
$f(x)$				

(c) Sketch a graph of $f(x)$.

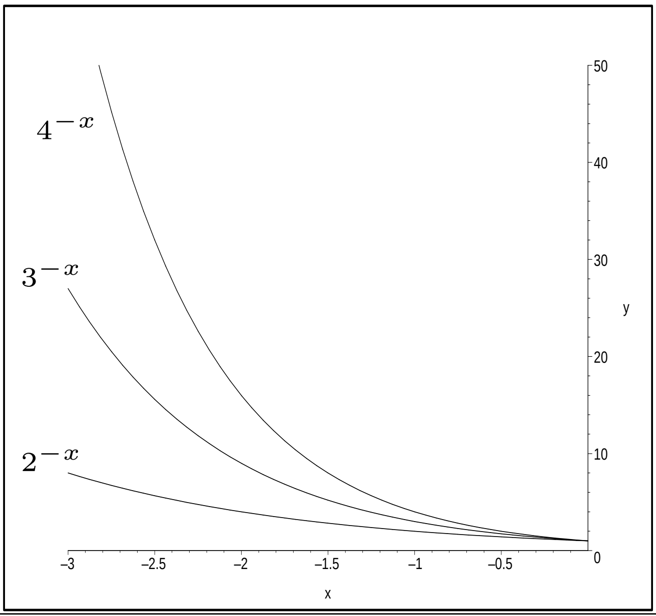


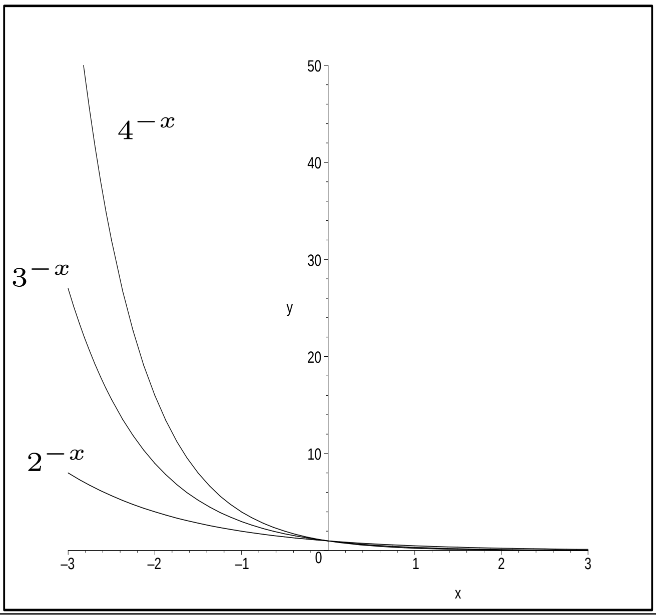
- The graph in Question 8.4.1 includes an exponential term, but y gets **smaller** as x gets bigger.
- Functions like this display **exponential decay**.
- Many quantities in business and nature decay exponentially.

Exponential decay functions.

If $a > 1$ and $C > 0$ then any function of the form $y = Ca^{-x}$ will display exponential decay.

***Example** 8.4.2* Plots of 2^{-x}, 3^{-x} and 4^{-x} for $x \in [-3, 0]$ (left) and $x \in [-3, 3]$ (right).





- When x is large and positive, y gets very close to 0 but never equals 0.
- When x is a large, negative number the graphs get very large and positive.
- The graphs decrease (decay) very quickly.
- The graphs all pass through the point $(0, 1)$.

8.5 The exponential function, e^x

- Any exponential growth function can be written as $y = Ca^x$ (where $a > 1$), and any exponential decay function can be written as $y = Ca^{-x}$, again with $a > 1$.
- Is any choice of value for a 'better' than other choices?

***Example** 8.5.1* You borrow \$1000 from a loan shark, who charges 100% interest for a month. What is the amount A you owe at the end of the month if interest is compounded:

- each month?

  Answer: In each case, $A = P(1+r)^n$, where $P = 1000$, n is the number of time periods and r is the interest rate per time period, expressed as a decimal number.
  Here, $n = 1$ and $r = 1$, so $A = 1000(2)^1 = \$2000$.

- each day? (Assume 30 days in the month.)

  Answer: Here, $n = 30$ and r=1/30, so $A = 1000(1 + 1/30)^{30} = \$2674.32$.

- each hour?

  Answer: Here, $n = 30 \times 24 = 720$ and r=1/720, so $A = 1000(1 + 1/720)^{720} = \$2716.40$.

- each minute? Answer: $A = \$2718.25$

- each second? Answer: $A = \$2718.28$

It seems likely that as the time for compounding gets shorter and shorter, the final amount will not keep getting indefinitely larger: it looks like we are approaching some fixed answer!

- In Example 8.5.1 we borrowed \$1,000, which turned into approximately \$2,718.28 when interest compounded very frequently at 100% for one time period.
- Dividing the final figure by 1000 (the amount initially borrowed), we see that each dollar we borrowed turned into \$2.71828...
- This number is so important that it has a special name.

The mathematical constant, e.

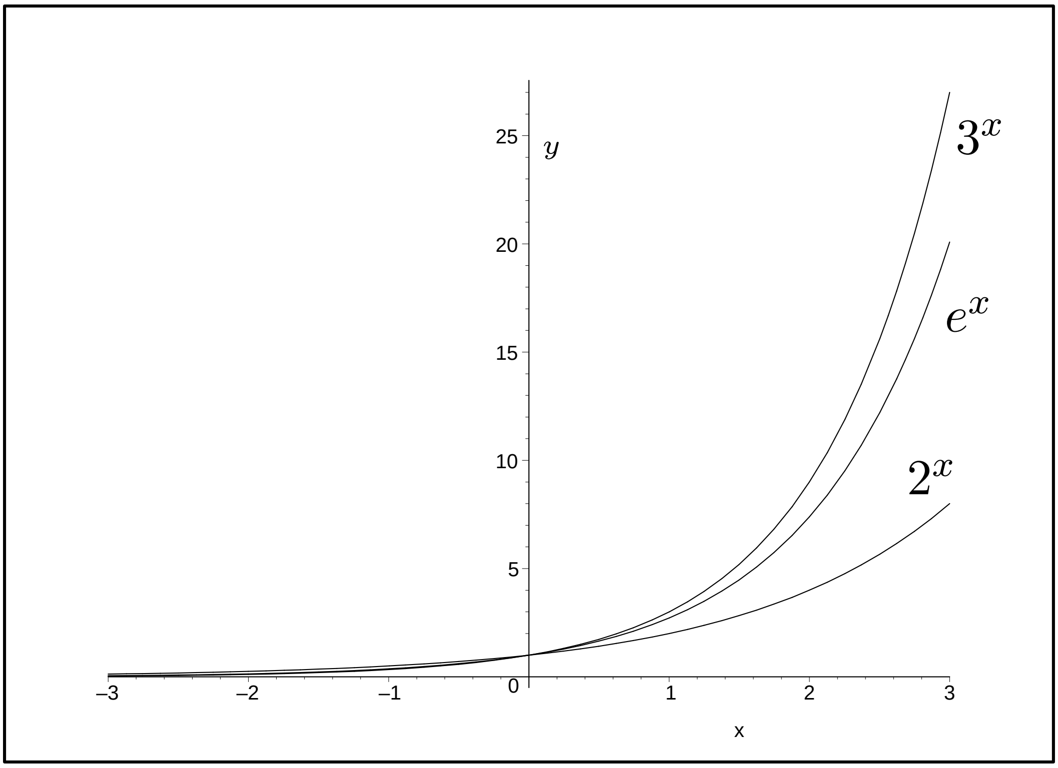
The mathematical constant e is an irrational number that arises in a large number of places. Because e is irrational, like π and $\sqrt{2}$, its value cannot be written exactly as a fraction or decimal. Its approximate value is:

$$e \approx 2.71828$$

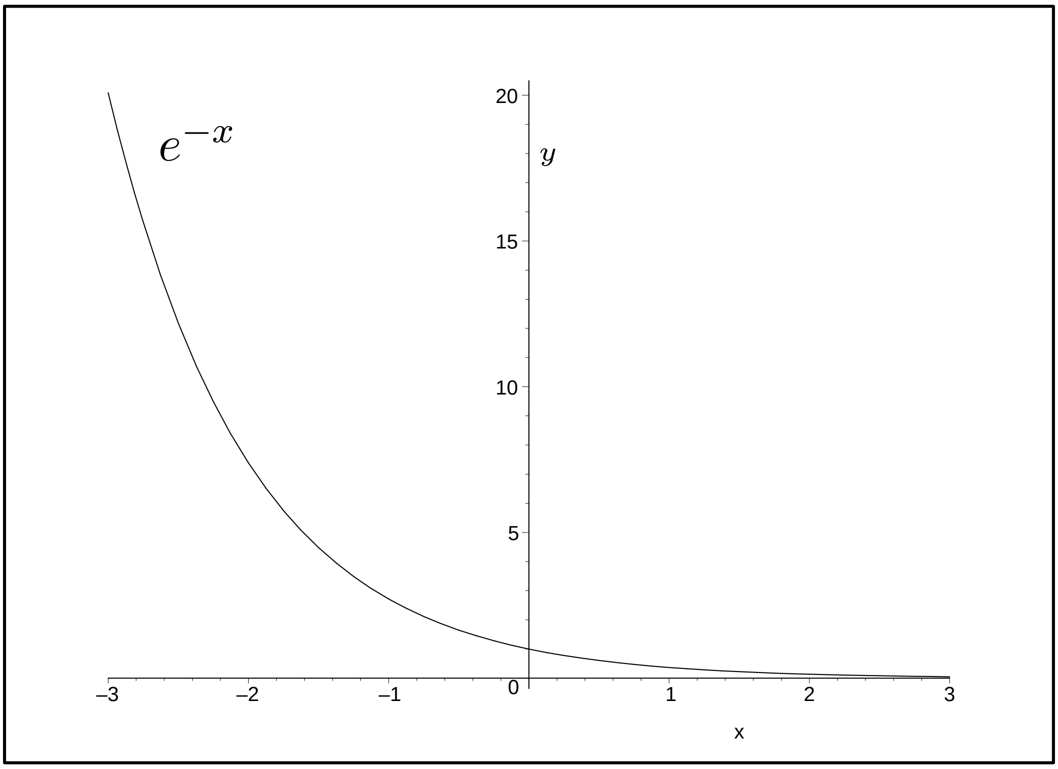
(This number is called e in honour of the famous 18th century Swiss mathematician Euler, one of the greatest mathematicians of all time.)

- e is the most frequently used base for exponential functions.
- There will probably be a key on your calculator labelled e^x (or possibly $exp(x)$).
- It may seem strange (and it certainly is complicated) to use an irrational number as the base: it is easy to calculate $2^2, 2^3, 2^4$ and so on, but much harder to calculate e^2, e^3, e^4.
- However, e occurs in many vital places, with interesting and useful properties, so is worth the extra effort.
- e is so important that we call e^x <u>the</u> exponential function.

- We can draw a plot of $f(x) = e^x$. Of course:
  - as $e > 1$, this graph will display exponential growth.
  - as $e > 2$ and $e < 3$, the graph will lie between 2^x and 3^x.
  - the graph will pass through $(0, 1)$.



Similarly, we know that e^{-x} must represent exponential decay:



- Because e^x is so important, from now on we will almost exclusively use e as the base.
- That is, given an exponential function with any base a, we can rewrite this as an equivalent function with a base of e and with the power changed in an appropriate way.
- Putting all of the above together, we have the following general form for exponential growth functions and exponential decay functions.

General form of exponential functions.

The general form of an exponential function is $y = Ce^{kx}$, where C and k are constants, $k \neq 0$ (and usually $C > 0$).

If k is positive the function gives exponential growth, and if k is negative the function gives exponential decay.

- Many things can all be specified by exponential functions involving e, including population growth and decline, growth in computer speed, the size of an economy and continuously compounding interest (in which the interest is continually added to an account all of the time).

***Example** 8.5.2* If \$P is invested at an annual interest rate of r, compounding **continuously**, then the amount of money at any time x is given by:

$$F(x) = Pe^{rx}.$$

***Example** 8.5.3* A certain radioactive substance decays exponentially. At time x the amount of material remaining is:

$$M(x) = Qe^{-kx},$$

where Q is the initial quantity and k is a constant depending on the substance.

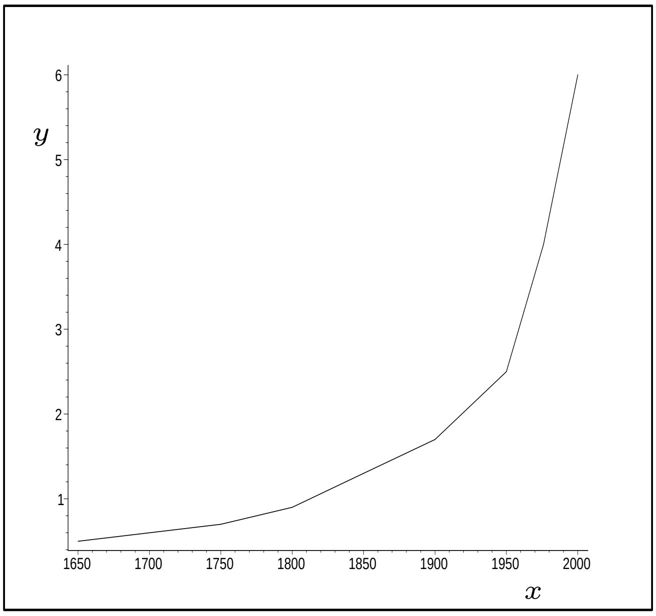
***Example** 8.5.4* The world's population is growing at about 2% per annum; assume this rate continues indefinitely.

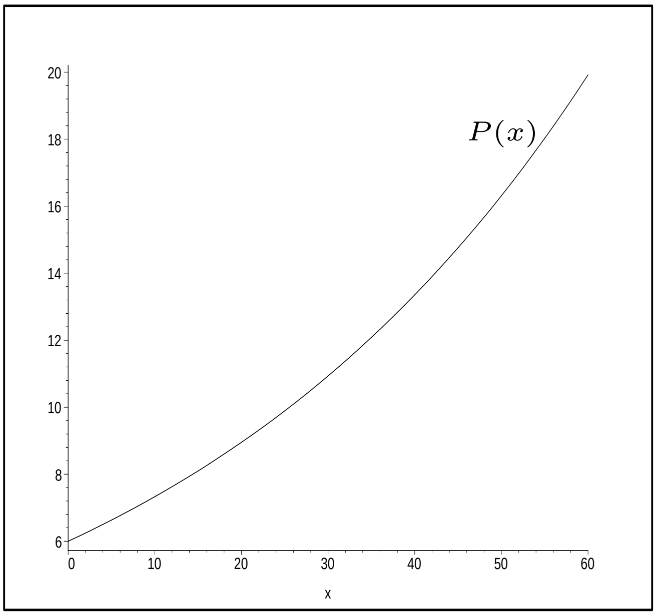
If the current population at time $x = 0$ is 6 billion, then the population at any time x will be approximately:

$$P(x) = 6000000000e^{0.02x}.$$

The left graph shows the world's population from 1650 until 2000. The right graph shows the above equation plotted from time 0 (the year 2000) until time 50 (the year 2050).

In each case, the y axis represents population in billions. Note the similarities in the shapes of the graphs.





8.6 Logarithms

- Given values for a and x, it is (often) not too hard to evaluate $y = a^x$. For example, if $a = 2$ and $x = 5$ then $a^x = 2^5 = 32$.
- Often, the reverse step is useful: given y and a, can we find x such that $y = a^x$? For example, if $81 = 3^x$ then how do we find x?
- Previously we have solved for x in linear and quadratic equations, but solving equations with x in the **power** requires a different technique.
- The reverse step (or *inverse function*) of an exponential function is called a *logarithm*.
- Sometimes the answer is fairly easy to find; in this course we'll only deal with this sort of question.

***Example** 8.6.1* Find x such that $1000 = 10^x$.

Answer: This question is asking: to what power should we raise 10 in order to get 1000?

Now $1000 = 10^3$ so $x = 3$.

***Question** 8.6.2* In each case, find x such that $y = a^x$:

(a) $y = 81$ and $a = 3$

(b) $y = 4$ and $a = 16$

Logarithms.

Let $a > 0, a \neq 1$ and $y = a^x$. Then we say that $x = log_a y$ (pronounced "x is the logarithm to base a of y").

***Question** 8.6.3* Rewrite your answers to Question 8.6.2 in logarithm form.

(a) Now $100 = 10^2$, so $\log_{10} 100 = 2$.

(b)

(c)

- We can take logarithms to any base $a > 0, \ a \neq 1$.
- The most common bases are 10 and e, giving log_{10} and log_e.
- Your calculators probably have keys for both of these.
- Don't be scared of logarithms: given a number y (eg $y = 10000$), $x = \log_{10} y$ simply means: x is the **power** to which we must raise the base (10) in order to get y. Because $10^4 = 10000$, we have $x = 4$.

Natural logarithms.

Logarithms to base e are called natural logarithms, *and the natural logarithm of x is written as $\ln x$.*

- Thus, given y, $x = \ln y$ simply means that x is the power to which we must raise e in order to get y.

***Question** 8.6.4* The intensity of earthquakes is measured using the Richter scale, which is a $\log_{10}$ scale. If an earthquake has intensity I, then its magnitude on the Richter scale is given by $R = log_{10}\left(\frac{I}{I_0}\right)$, where I_0 is a minimum intensity used for comparison (representing a very mild tremor).

(a) Rewrite this formula, making I the subject.

(b) The 2004 tsunami came from a quake measuring 8.9. How many times larger than I_0 is this? (Use $10^{0.9} = 7.943$.)

(c) In 1996, there was a magnitude 2.9 quake in Brisbane. How many times larger than this was the 2004 quake?

(d) An increase of 1 in the Richter magnitude results in 30 times more energy released. If magnitude 1 equals 10kg of TNT exploding, what does a magnitude 8.9 represent?

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9 Miscellaneous non-linear functions

Why are we covering this material?

- This is a very short section which discusses a few extra non-linear functions.
- We'll mostly see how to draw their graphs, and we'll briefly mention domain and range again.
- You will probably encounter these functions in your future work.
- We'll spend most time on circles, seeing how to find their equations.
- Circles will be useful when we study trigonometry.
- **Topics in this section are**
  - Non-linear functions.
  - Circles.

9.1 Non-linear functions

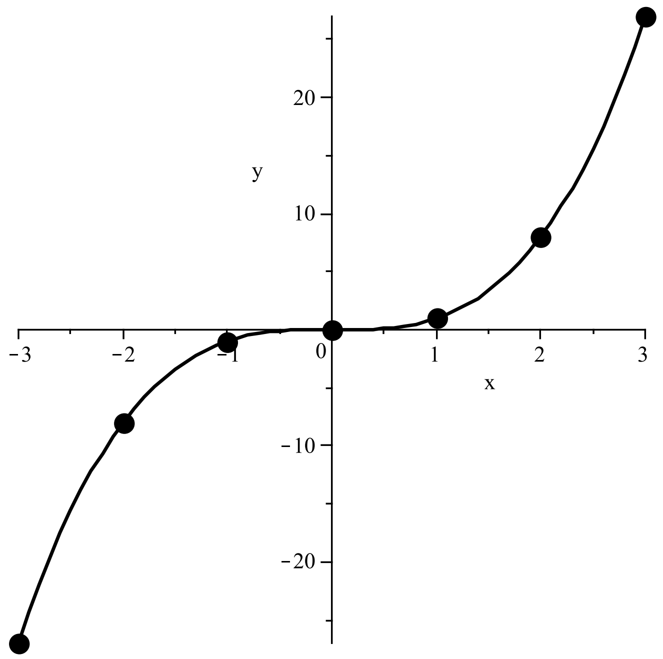
- We have seen graphs of linear equations, quadratic equations and exponential equations, but it is possible to draw the graphs of many other functions.
- You can still sketch many graphs by calculating a table of values, plotting the points and joining the dots.
- Note that you may need to use many data points in order to plot the graphs (not just two or three points).
- Also, always think about the equation you are plotting. Some points may not be in the domain.

***Example** 9.1.1* Sketch the graph of $y = x^3$.

First we create a table of values.

x	-3	-2	-1	0	1	2	3
y	-27	-8	-1	0	1	8	27

Finally, plot the graph:



***Example** 9.1.2* Sketch the graph of $y = \sqrt{x}$.

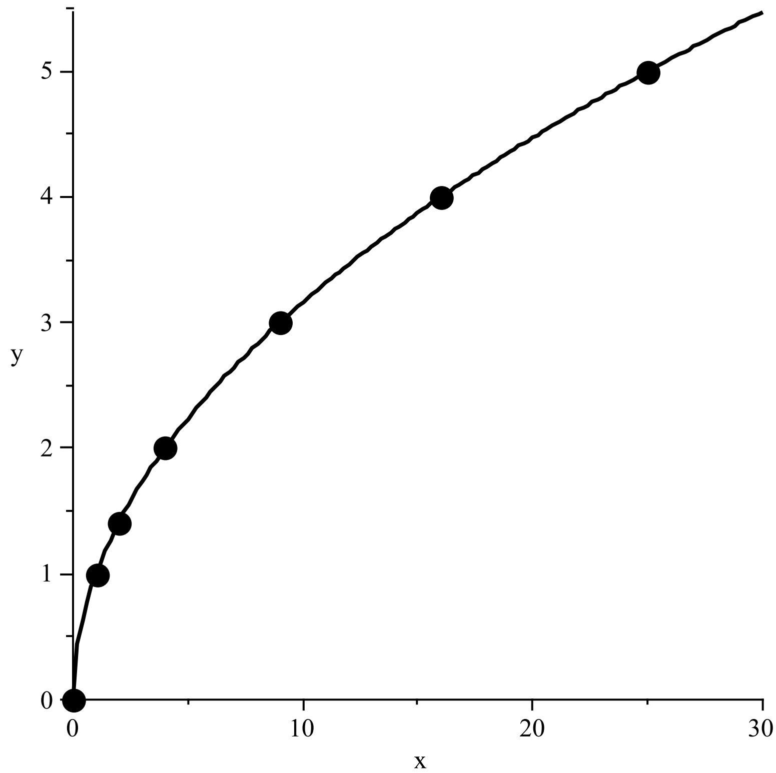
First, note that we cannot find the square root of a negative number. Hence the domain of $\sqrt{x}$ is $[0, \infty)$, so the graph should only exist for $x \geq 0$.

Also, when evaluating the function $y = \sqrt{x}$, it is always assumed that we are taking the positive square root. Hence the range of $\sqrt{x}$ is $[0, \infty)$, so the graph should only exist for $y \geq 0$.

Create a table of values:

x	0	1	2	4	9	16	25
y	0	1	$1.41421\ldots$	2	3	4	5

Finally, plot the graph:



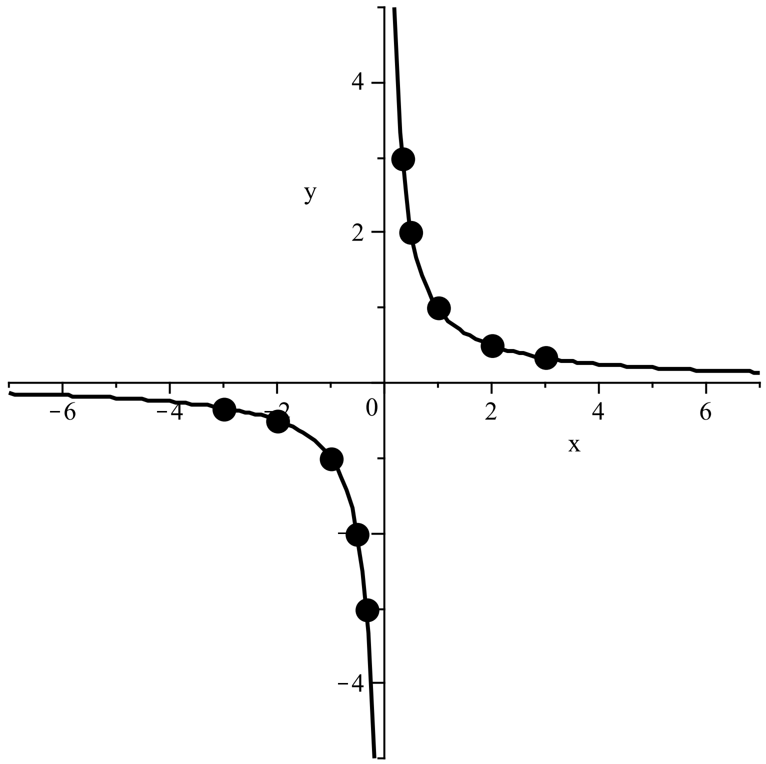
Example 9.1.3 Sketch the graph of $y = \dfrac{1}{x}$.

First, note that we cannot divide by 0. Hence the domain of $1/x$ is $(-\infty, 0) \cup (0, \infty)$, so the graph should not exist at $x = 0$.

Also, we clearly cannot have $y = 0$, because $1/x = 0$ has no solution. Hence the graph cannot exist at $y = 0$, so the range must be $(-\infty, 0) \cup (0, \infty)$.

Create a table of values and draw the graph:

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$



There are lots of things to notice about this graph:

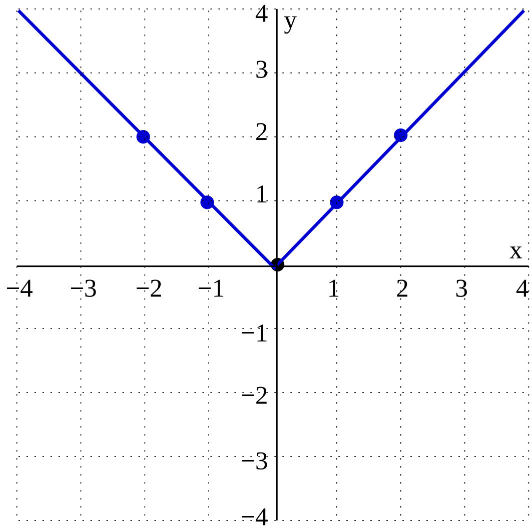
- The graph never touches or crosses the x-axis.
- The graph never touches or crosses the y-axis.
- As x gets big (and +ve or −ve), y gets close to 0.
- As x gets close to 0, y gets big (and positive if x is positive, or negative if x is negative).

Example *9.1.4* Sketch the graph of $y = |x|$.

Recall that $|x|$ means the absolute value of x, so the domain is $(-\infty, \infty)$ and the range is $[0, \infty)$.

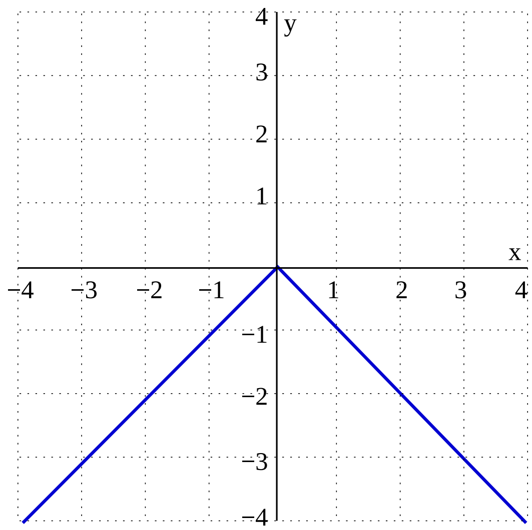
Create a table of values:

x	-2	-1	0	1	2
y	2	1	0	1	2



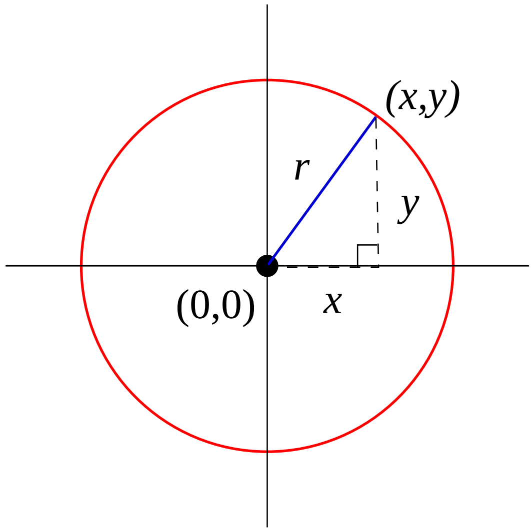
Example *9.1.5* Sketch the graph of $y = -|x|$.

This graph is easy to obtain from the graph in Example 9.1.4. In that example y was always positive. Here we have negated y, so it is always negative. Thus the domain is $(-\infty, \infty)$ and the range is $(-\infty, 0]$. The graph must be as follows:



9.2 Circles

- Just as we can give the mathematical equation of a straight line or a parabola, we can also give the mathematical equation of a circle.
- Consider a circle with centre $(0, 0)$ and radius r.
- Let (x, y) be any point on the circle, so (x, y) is a distance of r from the circle centre.
- We can draw a right-angled triangle as shown in the following diagram.



- Then Pythagoras' theorem gives us: $r^2 = x^2 + y^2$.
- This equation holds for **every point** on the circle.
- Hence we obtain the following.

Equation of a circle with centre $(0, 0)$.

A circle with centre at the origin and radius r has equation:

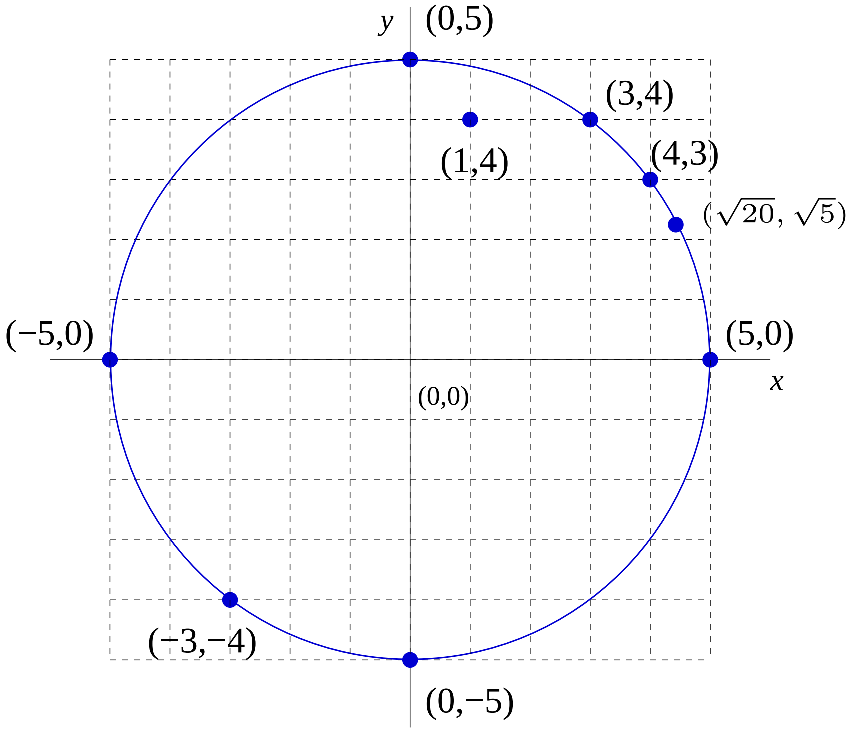
$$x^2 + y^2 = r^2.$$

Example *9.2.1* The circle with radius 5 centred at the origin has equation $x^2 + y^2 = 25$.

The point $(3, 4)$ lies on the circle; you can check this because $x^2 + y^2 = 3^2 + 4^2 = 25$ which is equal to r^2.

Similarly, $(4, 3)$, $(-3, -4)$, $(0, 5)$, $(0, -5)$, $(5, 0)$ $(-5, 0)$, and $(\sqrt{20}, \sqrt{5})$ all lie on this circle.

$(1, 4)$ does not lie on the circle, as $x^2 + y^2 = 1^2 + 4^2 = 17$.



Question *9.2.2* The *unit circle* is a circle of radius 1, centred at the origin. What is the equation of this circle, and where does it intersect the x and y axes?

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10 Trigonometry

Why are we covering this material?

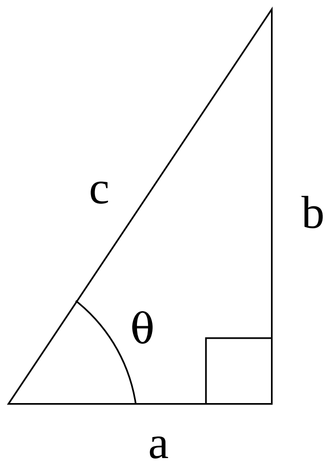
- Angles, degrees, radians, and the trig functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ form the basis of geometry, architecture, civil engineering, physics and surveying.
- Trig functions are fundamental to military applications, navigation and other methods of transport, including planes, boats and spacecraft.
- We'll see two ways of measuring angles: degrees and radians.
- We'll encounter trig functions and angles less than $90°$, then extend the concepts to angles greater than $90°$.
- We'll solve some practical problems, to see some uses for trig functions.
- We'll also examine the graphs of the trig functions and explain their appearance.
- **Topics in this section are**
  - Introduction to trigonometry.
  - More trigonometry.
  - Radians.
  - Angles bigger than $\pi/2$ ($90°$).
  - Graphs of the functions $\sin x$ and $\cos x$.

10.1 Introduction to trigonometry

- When studying angles, it's useful to define three trigonometric ratios.
- You should be familiar with these!

Trigonometric ratios.

In a right-angled triangle with hypotenuse of length c, other sides of lengths a and b, and a given angle θ, we can define the three trigonometric ratios $\sin\theta$, $\cos\theta$ *and* $\tan\theta$ *by:*



$$\sin\theta = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{b}{c}$$

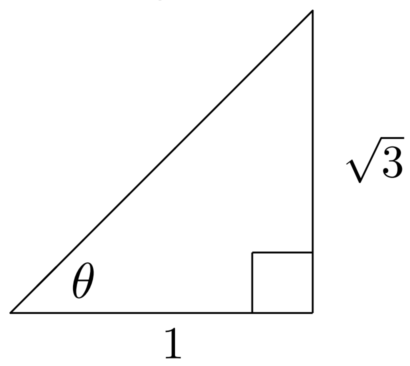
$$\cos\theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{a}{c}$$

$$\tan\theta = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}$$

***Question** 10.1.1* A 5 metre piece of wood is leant against a wall so that the angle between the ground and the wood is 30°. How far up the wall does the wood reach? (Hint: $\sin 30° = 0.5$.)

- Sometimes we know the lengths of two (or three) sides of a triangle but don't know any angles.
- We can use the trigonometric ratios to find angles.
- To do this, we need to find the *inverse* of the trig ratios.

***Example** 10.1.2* Find the angle θ in the following triangle:



We know the lengths of the adjacent and opposite sides for the angle θ, so we know that

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1}.$$

Thus, we need to find *the angle* θ *whose value of* tan *is* $\frac{\sqrt{3}}{1}$.

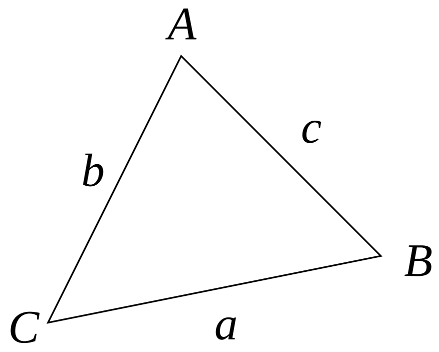
To do this we need to evaluate $\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$.

Using a calculator gives $\theta = 60°$.

- Note that "$\tan^{-1}$" is pronounced *inverse tan*, and does **not** mean $\frac{1}{\tan}$.
- Instead, $\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ means "the angle whose value of tan is $\frac{\sqrt{3}}{1}$".
- Similarly, $\sin^{-1}$ and $\cos^{-1}$ are the inverse functions of sin and cos respectively.
- The information in the question determines which one you should use.

10.2 More trigonometry

- When you work with triangles that are not right-angled triangles, the following two rules are useful.
- Let ABC be a triangle with side lengths a (opposite the angle at A), b (opposite the angle at B), and c (opposite the angle at C).

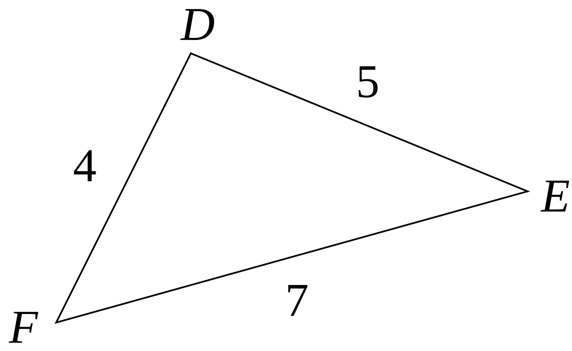


Cosine Rule $\quad a^2 = b^2 + c^2 - 2bc\cos A$

Sine Rule $\quad \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

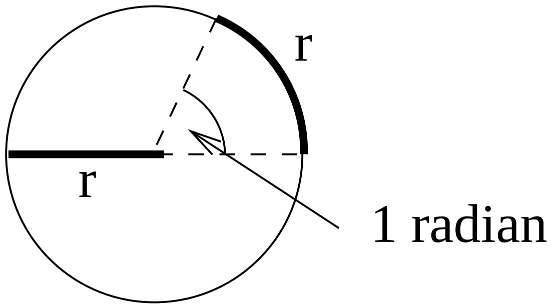
- Although the Sine Rule is usually quicker to apply than the Cosine Rule, you have to be careful with the Sine Rule if your triangle contains an obtuse angle (greater than a right angle).

Question *10.2.1* Determine the angles (in degrees) in the following triangle.



10.3 Radians

- A degree is a unit for measuring angles.
- A different unit for measuring angles is *radians*.
- In a circle of radius r, one radian is the angle which leads to an arc of length r on the circumference of the circle.



- We can easily convert from degrees to radians, and from radians to degrees.
- A circle of radius r has circumference $2\pi r$, so there are 2π radians in any circle. There are 360° in any circle. Hence:

Degrees → radians and radians → degrees.

- *To convert an angle from degrees to radians, multiply the angle by* $2\pi/360 \quad (= \pi/180)$.
- *To convert an angle from radians to degrees, multiply the angle by* $360/(2\pi) \quad (= 180/\pi)$.

***Example** 10.3.1* Converting angles from degrees to radians:

$$360^\circ \times \frac{\pi}{180} = 2\pi \text{ radians} \qquad 180^\circ \times \frac{\pi}{180} = \pi \text{ radians}$$

$$90^\circ \times \frac{\pi}{180} = \pi/2 \text{ radians} \qquad 45^\circ \times \frac{\pi}{180} = \pi/4 \text{ radians}$$

***Example** 10.3.2* Converting angles from radians to degrees:

$$\frac{\pi}{3} \text{ radians } \times \frac{180}{\pi} = 60^\circ \qquad \frac{\pi}{6} \text{ radians } \times \frac{180}{\pi} = 30^\circ$$

$$\frac{5\pi}{3} \text{ radians } \times \frac{180}{\pi} = 300^\circ \qquad 1 \text{ radian } \times \frac{180}{\pi} \approx 57.3^\circ$$

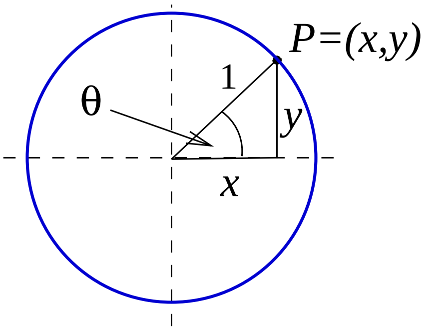
10.4 Angles bigger than $\pi/2$ (90°)

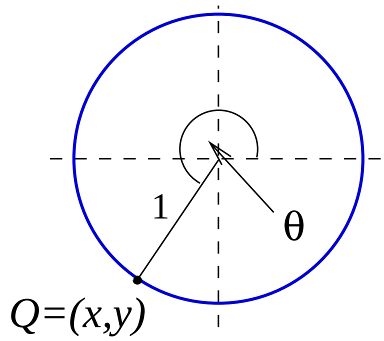
- Thus far, we have mostly looked at angles between 0 and 90° (equivalently, between 0 and $\pi/2$ radians).
- The trig ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$ have only been defined for angles in this range (as it is not possible to have an angle larger than 90° in a right-angled triangle).
- Of course, it is possible to have angles larger than $\pi/2$ (remember, these are called *obtuse* angles).
- By convention, angles are measured in the x, y plane, anti-clockwise around from the positive x-axis. Negative angles are measured clockwise from the positive x-axis.

***Question** 10.4.1* Mark each of the following angles on a set of axes: $135^\circ, 180^\circ, 240^\circ, 360^\circ, 450^\circ, -30^\circ$.

We need to extend the definitions of the trig functions so they also apply to angles more than 90°. Our definitions are based on the unit circle (with radius 1 centred on the origin; see Page 187) and are completely consistent with right-angled triangles.

- Draw a right-angled triangle from the origin, to the point $(x, 0)$, to the point $P = (x, y)$ and back to the origin. Call the angle at the origin θ. (See below, left.)





- The lengths of the three sides of the triangle are x (the horizontal side); y (the vertical side); and 1 (the hypotenuse; it is the radius of the circle).

- Using the right-angled triangle: $\cos\theta = \dfrac{x}{1} = x$, $\sin\theta = \dfrac{y}{1} = y$, and $\tan\theta = \dfrac{y}{x}$.

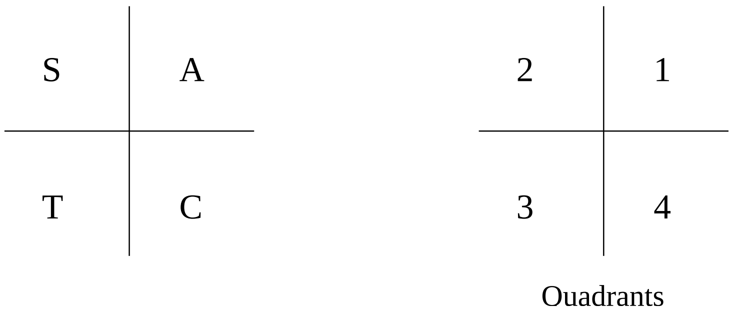
- Thus we can work out the trig values for the angle θ simply by reading the coordinates of the point $P = (x, y)$. We don't need to use the triangle or take ratios at all.

- Now we extend this approach to *any* point on the circle, not just in the first quadrant; see the right-hand diagram.

- Given *any* point $Q = (x, y)$ on the circle, we can draw the angle θ as above, obtaining any angle between 0° and 360°, and we have $\sin\theta = y$ and $\cos\theta = x$.

- This gives trig values for any angle: simply read the (x, y) coordinates of the corresponding point on the unit circle.

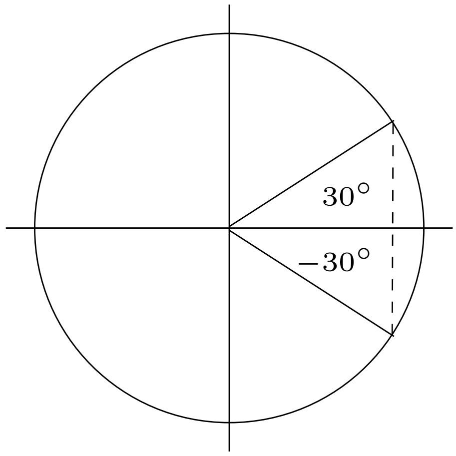
- Clearly:
  - $\cos\theta$ will be positive whenever the x-coordinate is positive, and negative whenever x is negative.
  - $\sin\theta$ will be positive whenever y is positive, and negative whenever y is negative.
- There is a quick way to remember which trig values are positive in which quadrants. It's called the "CAST" method, and is illustrated in the following diagram:



  - 'C' in Quadrant 4 means that for angles θ here, only $\cos\theta$ is positive.
  - 'A' in Quadrant 1 means that for angles θ here, *all* trig functions ($\sin\theta$, $\cos\theta$ and $\tan\theta$) are positive.
  - 'S' in Quadrant 2 means that for angles θ here, only $\sin\theta$ is positive.
  - 'T' in Quadrant 3 means that for angles θ here, only $\tan\theta$ is positive.
- If you think about angles in different quadrants you'll see that there are many relationships between trig ratios in different quadrants.
- For example, $\sin x = \sin(180° - x)$.
- From our CAST diagram, we can see that there are **two** angles for each value of sin, cos and tan.

Example *10.4.2* Find all angles θ between 0 and 360° which have the same cos value as does 30°.

Recalling that the *cos* value of an angle is the x-coordinate of the corresponding point on the unit circle, there is an angle in the fourth quadrant with the same x-coordinate. We can see this by drawing a vertical line through the x-axis to the circumference of the circle:



Hence the other angle is −30°, which is the same as 330°.

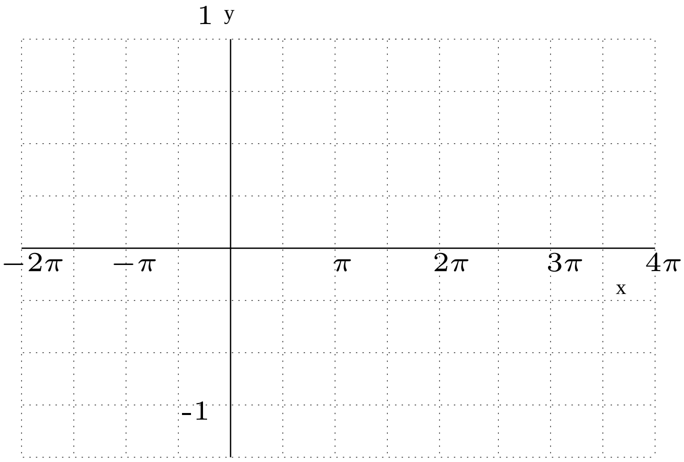
- When asked to find **all** angles from 0 to 360°, make sure you find them all!

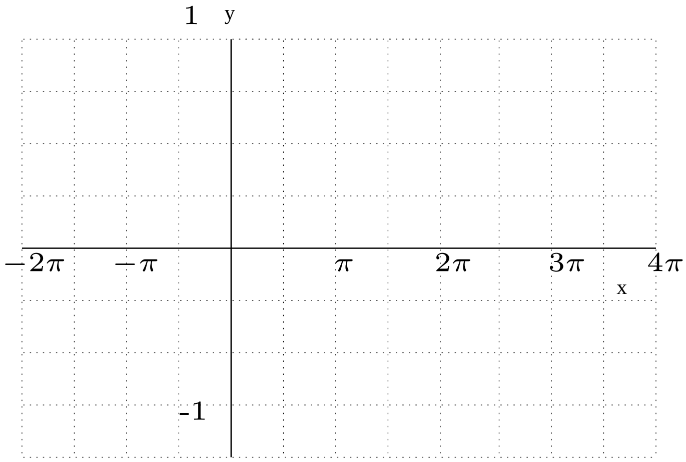
Question *10.4.3* Solve $5 = 4 + 2\sin\theta$, where $0 \le \theta \le 360°$.

10.5 Graphs of $\sin x$ and $\cos x$

- We've defined $\sin\theta$, $\cos\theta$ and $\tan\theta$ for any angles.
- Thus $\sin x$, $\cos x$ and $\tan x$ can be thought of as functions of the given angle x, and we can draw their graphs by calculating a lot of points.
- Graphs are usually plotted with x in radians.
- Because $\cos x$ and $\sin x$ are defined by the x- and y-coordinates (respectively) of points on the unit circle:
  - The range of both $\cos x$ and $\sin x$ is $[-1, 1]$
  - The graphs must 'repeat' every 2π radians (360°).
  - An angle of 0 radians (0°) corresponds to the point $(1, 0)$ on the unit circle, so $\cos 0 = 1$ and $\sin 0 = 0$.
  - An angle of $\pi/2$ radians (90°) corresponds to the point $(0, 1)$ on the unit circle, so $\cos\pi/2 = 0$ and $\sin\pi/2 = 1$.
  - An angle of π radians (180°) corresponds to the point $(-1, 0)$ on the unit circle, so $\cos\pi = -1$ and $\sin\pi = 0$.
  - An angle of $3\pi/2$ (270°) corresponds to the point $(0, -1)$ on the unit circle, so $\cos 3\pi/2 = 0$ and $\sin 3\pi/2 = -1$.

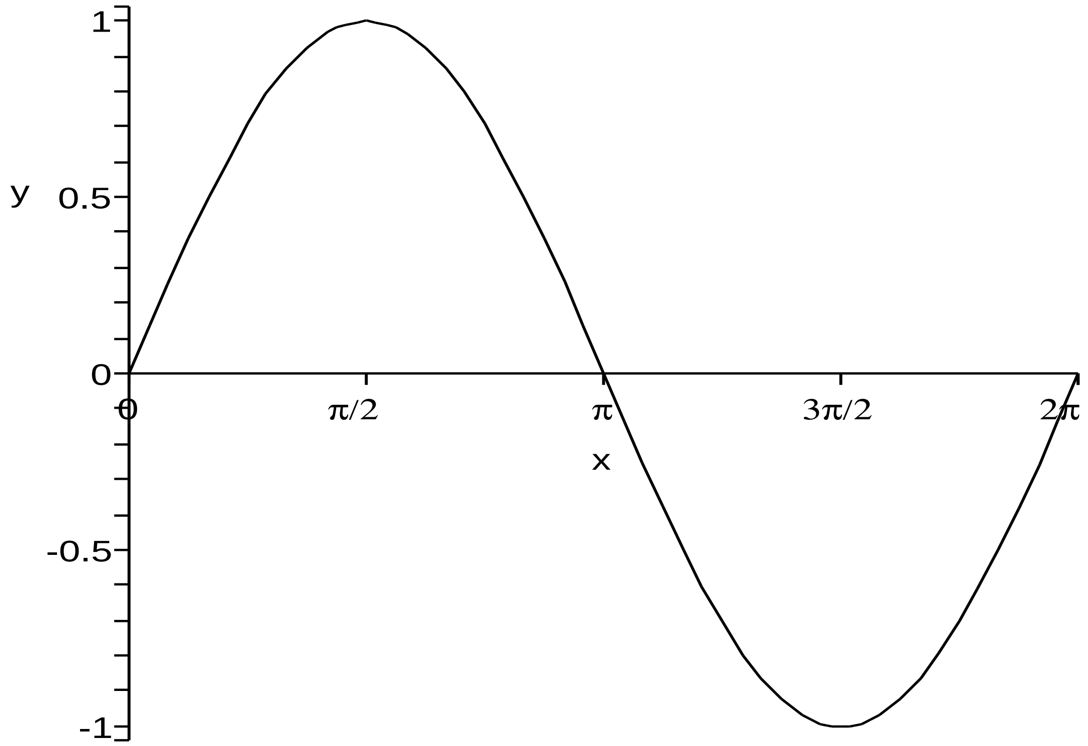
***Question** 10.5.1* Use the above data to plot some points on the following axes, with $\sin x$ on the left, and $\cos x$ on the right. Use $x \in [-2\pi, 4\pi]$.



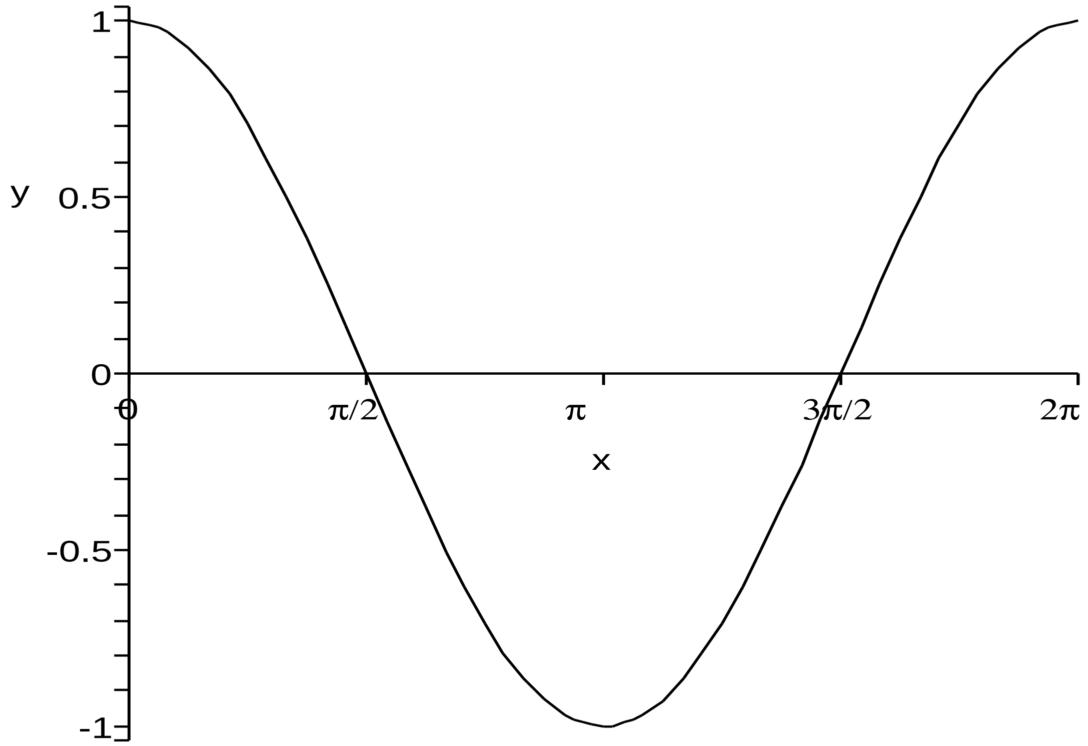


Here are accurate plots of $\sin x$ and $\cos x$ for $x \in [0, 2\pi]$.

Graph of $\sin x$ on $x \in [0, 2\pi]$:

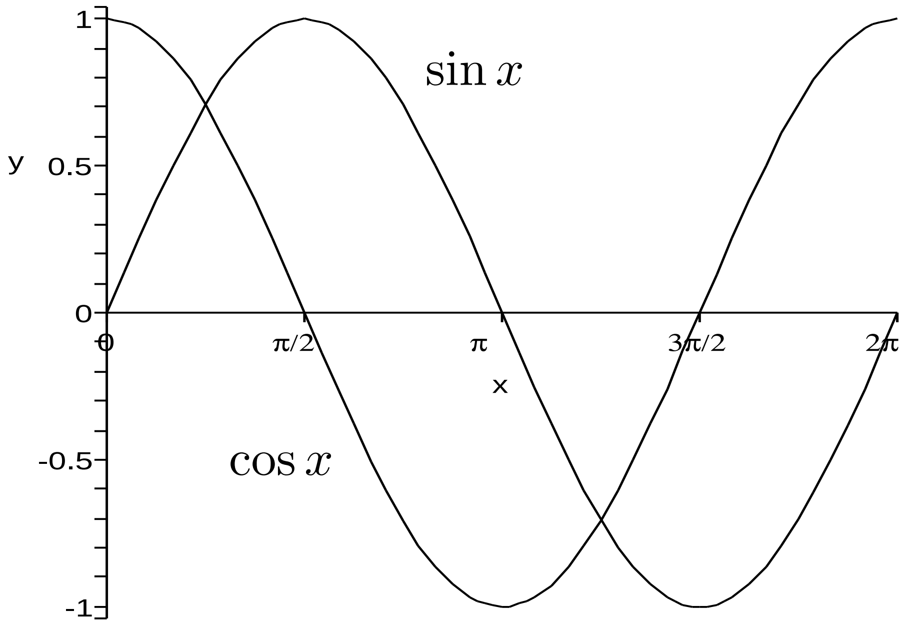


Graph of $\cos x$ on $x \in [0, 2\pi]$:



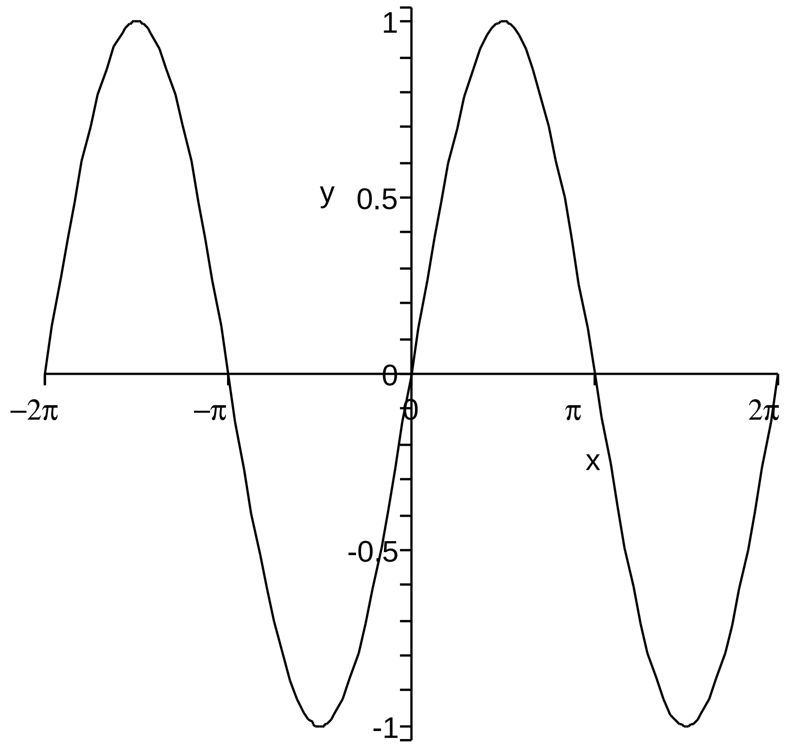
- For comparison we can plot $\sin x$ and $\cos x$ on one graph:

Graph of $\sin x$ and $\cos x$ on $x \in [0, 2\pi]$:



- We have said that the trig graphs must repeat every 2π radians (360°). Hence we have the following graph of $\sin x$ as x goes from -2π to 2π ($-360°$ to $360°$).

Graph of $\sin x$ on $x \in [-2\pi, 2\pi]$:



- Let a and b be numbers, and consider functions like $f_1(x) = a \sin x$, $f_2(x) = \sin bx$ and $f_3(x) = a \sin bx$. (The same variations are possible for $\cos x$.)
- Each of these is a smooth repeating wave, like the original functions. However, they differ from the originals in:
  - their **amplitude**, which is the vertical distance from the centre to the lowest (or highest) point in each 'wave'; and/or
  - their **frequency**, which is the number of complete waves in a given length.

Changing amplitude/frequency of trig functions.

The graph of '$a \sin bx$' has an amplitude that is a times that of $\sin x$ and a frequency that is b times that of $\sin x$.

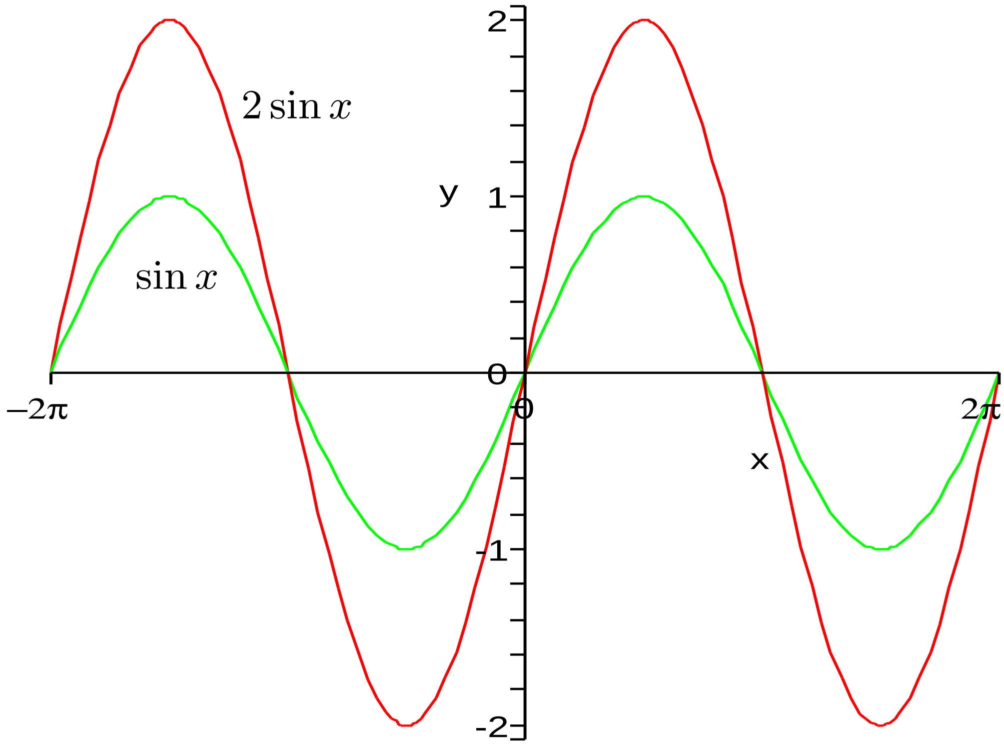
***Example** 10.5.2* The graph of $\sin x$ has amplitude 1 (its range is $[-1, 1]$) and repeats every 2π radians.

The graph of $3 \sin x$ has amplitude 3 (its range is $[-3, 3]$) and repeats every 2π radians.

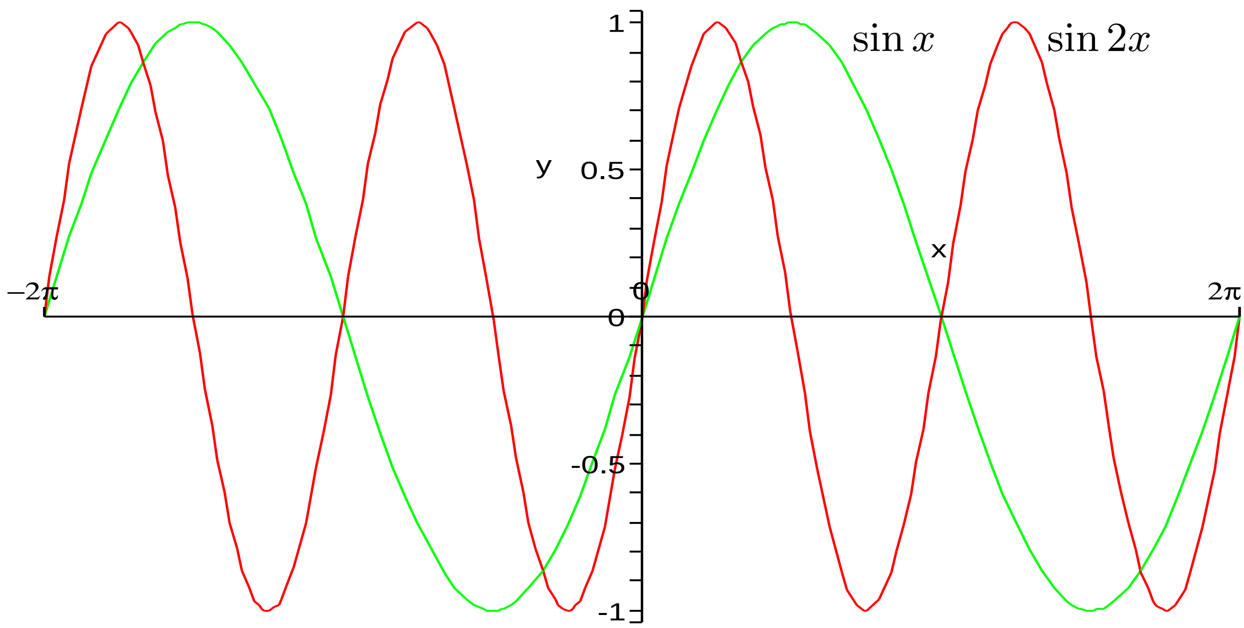
The graph of $\sin 2x$ has amplitude 1 (its range is $[-1, 1]$) and repeats every π radians (so has twice the frequency of $\sin x$).

The graph of $5 \sin 2x$ has amplitude 5 (its range is $[-5, 5]$) and repeats every π radians (so has twice the frequency of $\sin x$).

Graph of $\sin x$ and $2\sin x$ on $x \in [-2\pi, 2\pi]$:



Graph of $\sin x$ and $\sin 2x$ on $x \in [-2\pi, 2\pi]$:



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11 Derivatives and rates of change

Why are we covering this material?

- Now we start work on *calculus*.
- We will see how to find the rate of change (or *slope*) of an arbitrary graph.
- This is one of the most useful applications of mathematics. For example, marginal cost and marginal revenue curves from economics rely on it.
- We'll solve a number of simple differentiation problems, then encounter some rules which allow more complex functions to be differentiated.
- Pay careful attention to all of this material. Most of what we do from now on hinges critically on this. If you get lost, you'll be in trouble for the rest of the semester!
- It looks different to what we have seen before, and might be a bit confusing at first, but it's really not hard at all.
- **Topics in this section are**
  - Differentiation and derivatives.
  - Interpreting derivatives.
  - Simple differentiation.
  - Derivatives of some common functions.
  - Product rule.
  - Quotient rule.
  - Chain rule.
  - Second derivatives.

11.1 Differentiation and derivatives

- In a straight line $y = mx + c$, m is the gradient or slope of the line, which is constant over the whole line.
- It is incredibly useful to look at the slopes of graphs which are not straight lines.
- Obviously, in such graphs the slope is not the same over the entire graph: it will vary from point to point.
- There are special names and notations when finding slopes.

Differentiation.

- *The* **derivative** *of a function $f(x)$ gives the* **slope** *of $f(x)$ at any point, and the process of finding the derivative is called* **differentiation**.
- *The derivative of $f(x)$ can be written in two ways:*

(a) $f'(x)$, *pronounced "f dashed x"; or*

(b) $\dfrac{df}{dx}$, *pronounced "df dx".*

- *The value of the derivative at the point $x = a$ gives the slope of $f(x)$ at the point $x = a$.*

- Sometimes, derivatives do not exist at certain points. For example, the graph of $y = 1/x$ does not exist at $x = 0$, so it doesn't make sense to talk about its slope at that point. Here, we'll assume that derivatives exist whenever needed.
- The notation $\dfrac{df}{dx}$ is **not** a fraction; it is a concept. The d's do not cancel, and it is not df divided by dx. The f means that f is the function being differentiated, and the x means that differentiation is done with respect to x (written "w.r.t. x").

11.2 Interpreting derivatives

- Before we see how to find derivatives, it's important that you understand what they mean.

The meaning of derivatives.

The derivative of a function is itself a function, whose **value** *at any point x gives the* **slope** *of the original function at that point x.*

- Be quite clear on the distinction between:
  - the **value** of the function at a point; and
  - the **slope** of the function at a point.
- The value of the function at a point is the y-coordinate of the point, which can be found by substituting the x-coordinate into the expression for the **function**.
- The slope of the function at a point is a measure of how quickly the function is changing at that point. This can be found by substituting the x-coordinate into the expression for the **derivative**.
- The slope of the function at a point is equal to the value of the derivative at the x-coordinate of that point.
- Because the original function and its derivative are both functions, they can both be plotted on a set of axes.
- You should be able to look at the function and its derivative, and explain relationships between them.
- For example, if the derivative is negative at a certain point then the function must have negative slope at that point, so the function should be getting smaller.

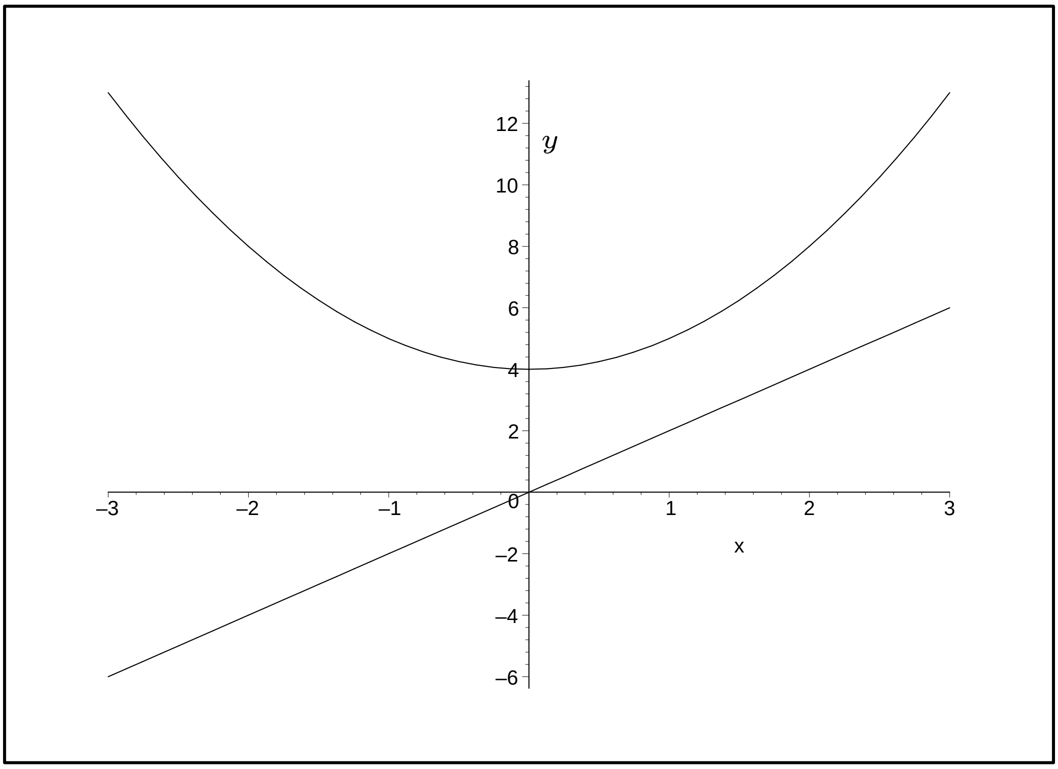
Question *11.2.1* The following figure shows the function $f(x) = x^2 + 4$, and its derivative $f'(x) = 2x$. Identify which graph is the function and which is the derivative. Then observe the relationships between f and f' when:

(1) f' is 0.

(2) f' is positive.

(3) f' is negative.

(4) f' changes from negative to positive.



Question *11.2.2* The following figure shows the function $f(x) = x^3 - 12x + 5$, and its derivative $f'(x) = 3x^2 - 12$. Identify which is which. Then observe the relationships between f and f' when:

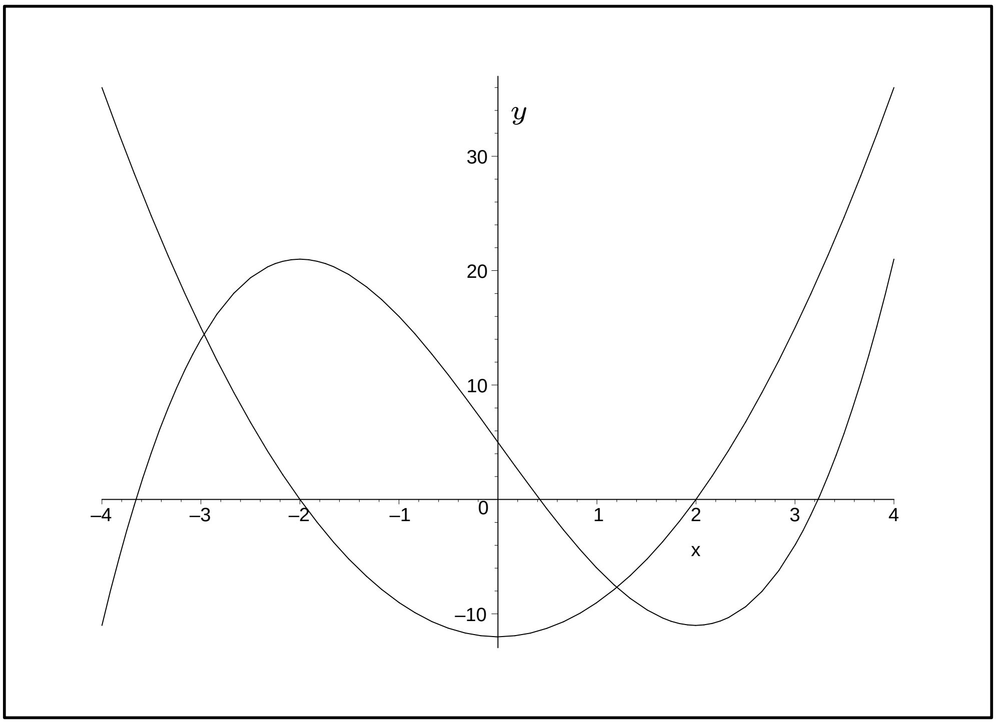
(1) f' is 0.

(2) f' is positive.

(3) f' is negative.

(4) f' changes from positive to negative.

(5) f' changes from negative to positive.



Question *11.2.3* The following figure shows the function $f(x) = \sin x$, and its derivative $f'(x) = \cos x$, on $x \in [0, 2\pi]$. Identify which is which. Then explain the relationships between f and f' when:

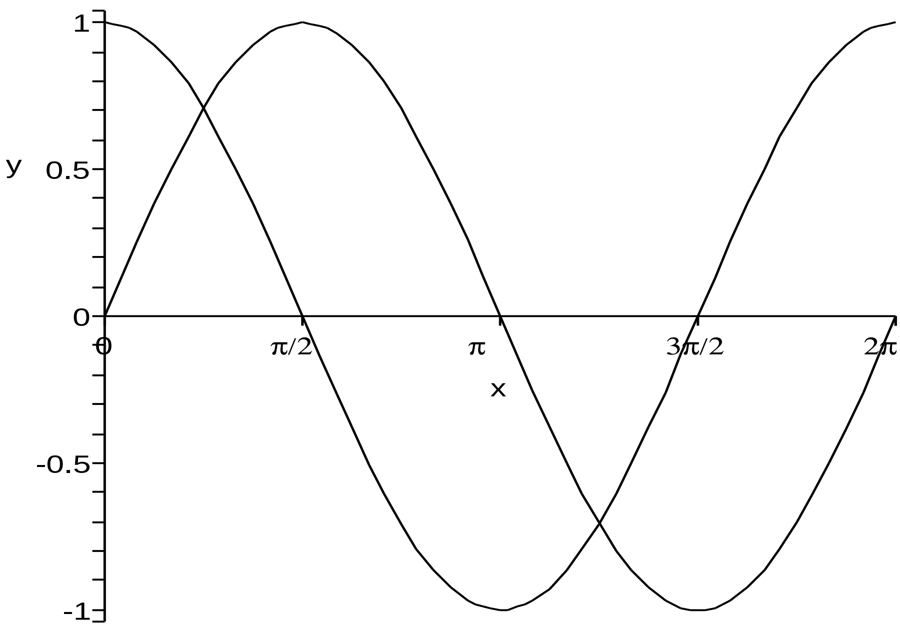
(1) f' is 0.

(2) f' is positive.

(3) f' is negative.

(4) f' changes from positive to negative.

(5) f' changes from negative to positive.



11.3 Simple differentiation

- There are a number of useful rules which allow derivatives to be calculated surprisingly easily.
- We'll cover differentiation rules in this section and in the next two sections.
- The first collection of rules is summarised in the following table.

Simple differentiation.

Let $f(x)$ and $g(x)$ be functions. Then:

- *If n is any non-zero number and $f(x) = x^n$ then $f'(x) = nx^{n-1}$.*
- *If c is any constant and $f(x) = c$ then $f'(x) = 0$.*
- *If c is any constant then $(cf)' = cf'$.*
- $(f + g)' = f' + g'$.
- $(f - g)' = f' - g'$.

- These rules are described in detail on the next few pages, with some examples.

Let n be any non-zero number. If $y = x^n$ then $y' = nx^{n-1}$.

***Example** 11.3.1* Examples of using this rule include:

- If $y = x^2$ then $n = 2$, so $y' = 2 \times x^{2-1} = 2x^1 = 2x$.
- If $y = x^3$ then $n = 3$, so $y' = 3 \times x^{3-1} = 3x^2$.
- If $y = x$ then $n = 1$, so $y' = 1 \times x^{1-1} = 1x^0 = 1$.
- If $y = x^4$ then $y' = 4x^3$.
- If $y = \dfrac{1}{x}$ then $y = x^{-1}$ so $n = -1$.

  Hence $y' = -1 \times x^{-1-1} = -x^{-2} = -\dfrac{1}{x^2}$.
- If $y = x^{-2}$ then $y' = -2x^{-3} = \dfrac{-2}{x^3}$.
- If $y = \sqrt{x}$ then $y = x^{1/2}$ so $n = 1/2$.

  Hence $y' = \dfrac{1}{2}x^{1/2-1} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

***Question** 11.3.2* Find y' for each of the following:

(a) $y = x^{10}$

(b) $y = x^{-7}$

(c) $y = x^{3/2}$

The next rule involves a **constant multiplied by a function**: the derivative equals the constant times the derivative of the function.

Let c be any constant. Then $(cy)' = cy'$.

***Example** 11.3.3* Examples of using this rule include:

- If $y = 4x^2$ then $y' = 4\times2\times x^{2-1} = 8x^1 = 8x$.
- If $y = 5x^3$ then $y' = 5\times3\times x^{3-1} = 15x^2$.
- If $y = 7x$ then $y' = 7\times1\times x^{1-1} = 7x^0 = 7$.
- If $y = -6x^4$ then $y' = -24x^3$.
- If $y = \dfrac{x^9}{9}$ then $y' = x^8$.
- If $y = \dfrac{4}{x}$ then $y' = 4\times -1\times x^{-1-1} = -4x^{-2} = -\dfrac{4}{x^2}$.

***Question** 11.3.4* Find y' for each of the following:

(a) $y = 9x^3$

(b) $y = 8x^{1/2}$

(c) $y = 6x^{-5}$

The next rule involves the **sum** or **difference** of two functions: to find the derivative of the sum (or difference) of two functions, add (or subtract) the derivatives of each function.

If $f(x)$ and $g(x)$ are functions, then $(f+g)' = f' + g'$ and $(f-g)' = f' - g'$.

Example *11.3.5* Examples of using this rule include:

- If $y = x^2 + 3x$ then $y' = 2x + 3$.
- If $y = 3x^3 + 2x^2 + 4x + 1$ then $y' = 9x^2 + 4x + 4$.
- If $y = 6x + 6x^{-2}$ then $y' = 6 - 12x^{-3} = 6 - \dfrac{12}{x^3}$.
- If $y = x^2 - 3x$ then $y' = 2x - 3$.
- If $y = x^3 - 3x^2$ then $y' = 3x^2 - 6x$.

Question *11.3.6* Find y' for each of the following:

(a) $y = -2x^3 + 4x^2 + 3x - 5$

(b) $y = x^2 - x + \sqrt{x}$

(c) $y = 3 + 1/x$

11.4 Derivatives of some common functions

- In this section we'll see how to differentiate the functions $\sin x$, $\cos x$, e^x and $\ln x$.

Derivatives of trig functions.

If $y = \sin x$ *then* $y' = \cos x$.

If $y = \cos x$ *then* $y' = -\sin x$.

Example *11.4.1* Examples of using this rule include:

- If $y = 3\sin x$ then $y' = 3\cos x$.
- If $y = -\sin x$ then $y' = -\cos x$.
- If $y = -\cos x$ then $y' = \sin x$.

Question *11.4.2* Find y' for each of the following:

(a) $y = x^2 + \sin x$

(b) $y = -x^3 + 4x + 4\sin x$

(c) $y = x^2 - \cos x$

(d) $y = \sin x + 2\cos x$

- One of the most important properties of the exponential function e^x relates to its derivative.
- At any point on the graph of $y = e^x$, the **slope** of the graph **equals the value** of the graph.
- e^x is the unique (non-zero) function for which this is true (that is, no other non-zero function satisfies this property).

Derivative of e^x.

If $y = e^x$ then $y' = e^x$.

- Think about what this means! When the function value is 1, its slope must also be 1. When function value is 2, the slope must be 2. When the function value is 37.8, the slope must also be 37.8, and so on.
- Thus, differentiating e^x is easy: it remains unchanged.
- Don't get mixed up with the rule that says if $y = x^n$ then $y' = nx^{n-1}$. That rule **only works** when we have x raised to a power which is a constant number.
- Here, we have e raised to a power which is a variable x.
- Thus the rule for differentiating e^x is completely different to the rule for differentiating x^n.
- In particular, when we differentiate e^x, **the power remains unchanged**.

***Example** 11.4.3* Examples of using this rule include:

- If $y = x^3 + e^x$ then $y' = 3x^2 + e^x$.
- If $y = \sin x - e^x$ then $y' = \cos x - e^x$.

Question *11.4.4* Find y' for each of the following:

(a) $y = e^x + 7$

(b) $y = 3x^2 + 2x + 4 + e^x$

- Finally, we see how to differentiate $\ln x$.

Differentiating $\ln x$.

If $y = \ln x$ then $y' = \dfrac{1}{x}$.

Example *11.4.5* Examples of using this rule include:

- If $y = \ln x + 7x + 6$ then $y' = \dfrac{1}{x} + 7$.
- If $y = 3x - 6\ln x$ then $y' = 3 - \dfrac{6}{x}$.

Question *11.4.6* Find y' for each of the following:

(a) $y = \ln x + \dfrac{1}{x}$

(b) $y = e^x - 3\ln x$

Summary of rules for differentiation.

Let $f(x)$ and $g(x)$ be functions. Then:

- *If n is any non-zero number and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.*
- *If c is any constant and $f(x) = c$, then $f'(x) = 0$.*
- *If c is any constant, then $(cf)' = cf'$.*
- $(f + g)' = f' + g'$.
- $(f - g)' = f' - g'$.
- *If $f(x) = \sin x$, then $f'(x) = \cos x$.*
- *If $f(x) = \cos x$, then $f'(x) = -\sin x$.*
- *If $f(x) = e^x$, then $f'(x) = e^x$.*
- *If $f(x) = \ln x$, then $f'(x) = \dfrac{1}{x}$.*

11.5 Product rule

- There are several additional important rules for finding derivatives.
- The first rule relates to finding the derivative of the **product** of two functions.
- In general, the answer does not simply equal the product of the derivatives!
- We need a special rule, called the *product rule*, which enables us to differentiate in this case.

Product rule.

Let $u(x)$ and $v(x)$ be functions. If $y(x) = u(x)\times v(x)$ then we differentiate y as follows:

$$y' = (uv)' = u'v + uv'$$

Another way of writing this is:

$$y' = \frac{dy}{dx} = \frac{du}{dx}\cdot v + u\cdot \frac{dv}{dx}$$

In English, the product rule says:

The derivative of the product of two functions equals

(the derivative of the first function) × (the second function) +

(the first function) × (the derivative of the second function)

***Example** 11.5.1* Here are some functions which must be differentiated using the product rule:

- $y = (3x + 4)\sin x$

  ($y = u\times v$ where $u = 3x + 4$ and $v = \sin x$)
- $y = (3x^2 + 2x + 4)(2x^4 - 9)$

  ($y = u\times v$ where $u = 3x^2 + 2x + 4$ and $v = 2x^4 - 9$)

***Example** 11.5.2* If $y = xe^x$, find y'.

Answer: Let $u = x$ and $v = e^x$, so $y = uv$.

$u' = 1$ and $v' = e^x$.

We then substitute into the product rule formula, so:

$$\begin{aligned} y' = u'v + uv' &= 1\times e^x + x\times e^x \\ &= e^x + xe^x \\ &= e^x(1 + x). \end{aligned}$$

Question *11.5.3* Let $f(x) = x^3 \times x^4$.

(a) Find $f'(x)$ by first simplifying f, then differentiating.

(b) Find $f'(x)$ using the product rule. Verify that your answer matches that in Part (a).

(c) Finally, verify that it is **not correct** to just let the derivative of the product be the product of the derivatives.

Example *11.5.4* Let $f(x) = x^2 \sin x$. Find $f'(x)$.
Let $u = x^2$ and $v = \sin x$, so $f(x) = uv$.
So $u' = 2x$ and $v' = \cos x$. Then:

$$\begin{aligned} f'(x) &= u'v + uv' \\ &= 2x \sin x + x^2 \cos x. \end{aligned}$$

Question *11.5.5* Let $y = x \ln x$. Find y'.

11.6 Quotient rule

- The next differentiation rule is called the *quotient rule*: it allows us to differentiate functions which are the **quotient** of two other functions.

Quotient rule.

Let $u(x)$ and $v(x)$ be functions. If $y(x) = \dfrac{u(x)}{v(x)}$ then we differentiate y as follows:

$$y' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Another way of writing this is:

$$y' = \frac{dy}{dx} = \frac{\frac{du}{dx} \cdot v - u \cdot \frac{dv}{dx}}{v^2}$$

***Example** 11.6.1* Here are some functions which must be differentiated using the quotient rule:

- $y = \dfrac{3x+1}{x^2}$ $\quad$ ($y = \dfrac{u}{v}$ where $u = 3x+1$ and $v = x^2$)
- $y = \dfrac{\sin x}{\cos x}$ $\quad$ ($y = \dfrac{u}{v}$ where $u = \sin x$ and $v = \cos x$)
- $y = \dfrac{e^x}{\ln x}$ $\quad$ ($y = \dfrac{u}{v}$ where $u = e^x$ and $v = \ln x$)

***Example** 11.6.2* If $y = \dfrac{x+1}{x-1}$, find y'.

Answer: Let $u(x) = x+1$ and $v(x) = x-1$, so $y = \dfrac{u}{v}$.

Clearly, $u' = 1$ and $v' = 1$. Then we substitute these (and u and v) into the quotient rule formula.

Then $y' = \dfrac{u'v - uv'}{v^2} = \dfrac{1 \times (x-1) - (x+1) \times 1}{(x-1)^2} =$

$$\frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

***Question** 11.6.3* Let $f(x) = \dfrac{(x+1)}{x}$

(a) Find $f'(x)$ by first simplifying f then differentiating.

(b) Find $f'(x)$ using the quotient rule. Verify that your answer matches that in Part (a).

11.7 Chain rule

- We have seen how to differentiate functions like $\sin x$, $\cos x$, e^x and $\ln x$.
- What if the **input** to the function is more complicated than just x?
- For example, how do we differentiate functions like e^{2x} or $\sin(3x+4)$?
- We need to introduce a new rule, called the *chain rule*.
- It's based on composition of functions, which we studied in Section 6. Some books call it the *composite function rule*.
- Most people find the chain rule to be the hardest of the differentiation rules.

Chain rule.

Let $u(x)$ be a function. If y is a function of $u(x)$, then

$$y' = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- The chain rule applies when we are differentiating a function y which is itself a function of another function $u(x)$.
- The chain rule says that to differentiate y with respect to x, we need to:
  - differentiate y with respect to u;
  - differentiate u with respect to x; and
  - multiply the two together.
- We need to be able to identify **when** to apply the chain rule, and **what** $u(x)$ is.

Example *11.7.1* Let *something* represent some function of x. The only functions we will encounter that require the chain rule to differentiate are:

- $y = (something)^{power}$ For example:
  1. $y = (2x+4)^2$, so $y = u^2$ where $u = (2x+4)$.
  2. $y = (1-x)^{15}$, so $y = u^{15}$ where $u = (1-x)$.
  3. $y = (\sin x)^7$, so $y = u^7$ where $u = \sin x$.

- $y = \sin(something)$ For example:
  1. $y = \sin(x+1)$, so $y = \sin u$ where $u = (x+1)$.
  2. $y = \sin(x^2)$, so $y = \sin u$ where $u = (x^2)$.

- $y = \cos(something)$ For example:
  1. $y = \cos(3x-4)$, so $y = \cos u$ where $u = (3x-4)$.
  2. $y = \cos(x-1)$, so $y = \cos u$ where $u = (x-1)$.

- $y = e^{(something)}$ For example:
  1. $y = e^{2x+4}$, so $y = e^u$ where $u = (2x+4)$.
  2. $y = e^{3x}$, so $y = e^u$ where $u = (3x)$.
  3. $y = e^{\sin x}$, so $y = e^u$ where $u = (\sin x)$.

- $y = \ln(something)$ For example:
  1. $y = \ln(2x)$, so $y = \ln u$ where $u = (2x)$.
  2. $y = \ln(x^2+2)$, so $y = \ln u$ where $u = (x^2+2)$.

- Recall that the chain rule says: if y is a function of $u(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
- The following example shows the chain rule in action.
- We are differentiating a particular function $y = (x^3 + 1)^2$ that certainly requires the chain rule to differentiate.
- However, we can also differentiate this function by first expanding it, then differentiating directly. We should get the same answer both ways.

***Example** 11.7.2* Let $y = (x^3 + 1)^2$

(a) Find $\frac{dy}{dx}$ using the chain rule.

Answer: Let $u = x^3 + 1$ so $y = u^2$.

Thus $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 3x^2$. From the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} . \frac{du}{dx} \\ &= 2u \times 3x^2 \\ &= 2(x^3 + 1) \times 3x^2 \\ &= 6x^2(x^3 + 1) \\ &= 6x^5 + 6x^2 \end{aligned}$$

(b) Find $\frac{dy}{dx}$ by first expanding y then differentiating.

Answer: $y = (x^3 + 1)^2 = (x^3 + 1)(x^3 + 1) = x^6 + 2x^3 + 1$,

so $\frac{dy}{dx} = 6x^5 + 6x^2$.

- There is an informal way of describing the chain rule.
- If this way helps, then you can think of the rule like this.
- You "work from the outside, going inwards".

***Example** 11.7.3* If $y = (x^3 + 1)^2$, find y'.

First, note that we have to use the chain rule.

Now, rewrite the function as $y = (\textit{something})^2$, where $\textit{something} = x^3 + 1$. We will worry about what to do with *something* later; for the moment, we are just looking at the outside bit of the function.

Then we differentiate $(\textit{something})^2$. If we were differentiating x^2, the answer would be $2 \times x$. Thus here the answer is $2 \times \textit{something}$.

Now we have finished with the outside bit of the function, so we look at the inside bit; that is, we look at $\textit{something} = x^3 + 1$. We need to differentiate it, so the derivative of $x^3 + 1$ is $3x^2$.

Finally, the chain rule says that we need to multiply both derivatives together, so

$$y' = 2 \times \textit{something} \times 3x^2 = 6x^2(x^3 + 1) = 6x^5 + 6x^2.$$

(Of course, if you think about what we have done above, then the *something* is simply the function $u(x)$, and all we are doing is following the exact steps as listed in the statement of the chain rule.)

Question *11.7.4* Find the derivative of $(3x^2+4)^{10}$.

Example *11.7.5* If $y=\sin(x^3)$, find y'.

Let $u=x^3$, so $y=\sin u$.

Thus $\dfrac{dy}{du}=\cos u$ and $\dfrac{du}{dx}=3x^2$. From the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\cos u\times 3x^2=\cos(x^3)\times 3x^2=3x^2\cos(x^3)$$

Question *11.7.6* Find the derivative of $\cos(3x^2+4)$.

Question *11.7.7* Let $y = 4e^{3x+2}$. Find y'.

Derivative of e^{kx}.

As a special case of the chain rule, if k is a constant and $y = e^{kx}$ then

$$y' = ke^{kx}$$

(Note that the power in e^{kx} remains unchanged.)

Example *11.7.8* If $y = e^{-x} + e^{4x}$ then $y' = -e^{-x} + 4e^{4x}$.

Question *11.7.9* Find the derivative of $e^{2x} + \ln(2x + 1)$.

Question *11.7.10* Find the derivative of $(e^{2x} + x^2)^7$.

11.8 Second derivatives

- Sometimes it's useful to find the derivative of the derivative.
- We use the same rules as we used to find the derivative.

Second derivatives.

Given a function $y = f(x)$, its derivative is written as

$$f'(x) \qquad \text{or} \qquad y' \qquad \text{or} \qquad \frac{dy}{dx}$$

We can find the derivative of the derivative, which is called the **second derivative**. *It is written as*

$$f''(x) \qquad \text{or} \qquad y'' \qquad \text{or} \qquad \frac{d^2y}{dx^2}$$

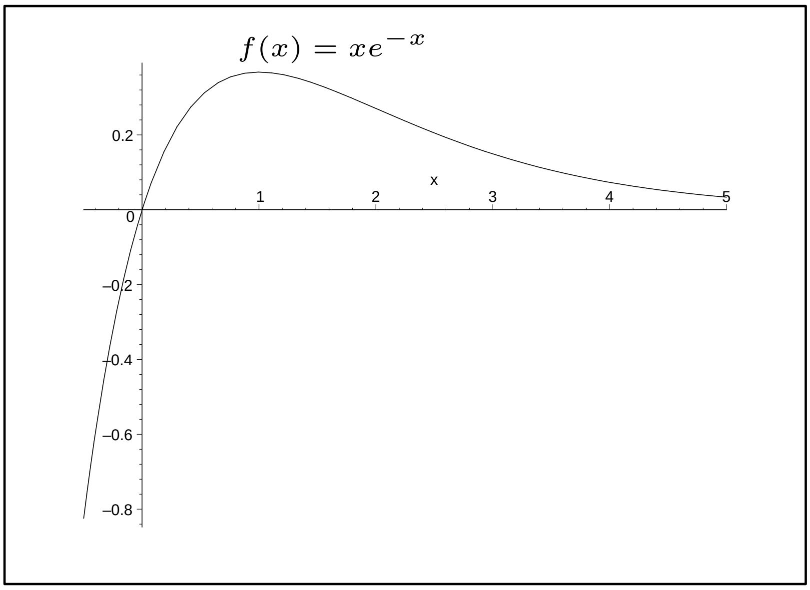
Example *11.8.1* If $f(x) = 16x^2 + 8x + 4$, find $f'(x)$ and $f''(x)$.
Answer: Clearly, $f'(x) = 32x + 8$ and $f''(x) = 32$.

Question *11.8.2* Find $f'(x)$ and $f''(x)$ if:

(a) $f(x) = x^3 - 6x^2 + 3x - 4\sin x$

(b) $f(x) = \ln x + e^{2x} + e^x$

Question *11.8.3* Let $f(x) = xe^{-x}$. (The graph is shown.)



(a) Find $f'(x)$.

(b) Solve $f'(x) = 0$.

(c) Find $f''(x)$.

NOTES

NOTES

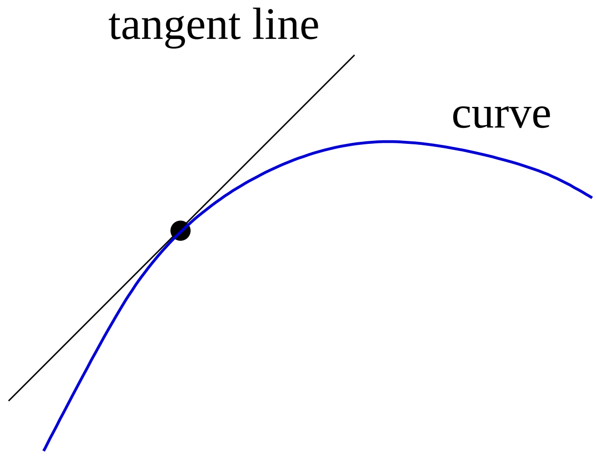
12 Applications of derivatives

Why are we covering this material?

- Knowing the derivative of a function gives a lot of useful information about the function, including:
  - where the function is increasing or decreasing.
  - where the function has peaks or troughs.
- This information can be used in a number of important practical ways, such as:
  - finding peaks and troughs allows the function value to be maximised or minimised, leading to optimal values.
  - solving production and economic (profit and cost) problems.
  - solving motion problems involving velocity, displacement and acceleration.
- We will study a number of applications. All arise directly from properties of the slope of a function.
- Make sure you are quite familiar with differentiation, as we will continually apply the skills and rules developed in Section 11.
- **Topics in this section are**
  - Tangent Lines.
  - Derivatives and motion.
  - Local maxima and minima.
  - Some practical problems.

12.1 Tangent Lines

- Given a curve, we can find a **tangent line** to the curve at a given point.
- A tangent line **touches** the curve at the given point, and has the **same slope** as the curve at that point.



Tangent lines

Given a function, we can find the tangent to the graph of that function at a given point by the following procedure:

1. *First, find the derivative of the function.*
2. *Next, find the slope of the curve at the given point, by substituting the given value of x* **into the derivative**.
3. *Finally, use the slope of the curve at that point as the gradient m in a straight line equation $y = mx + c$, and substitute the coordinates of the given point into this equation to give the value of c.*

- The last step should be familiar: you are finding the equation of a line given its gradient and a point on the line.

Example *12.1.1* Given $y = e^{0.5x}$, find the equation of the tangent line to y at the point $(0, 1)$.

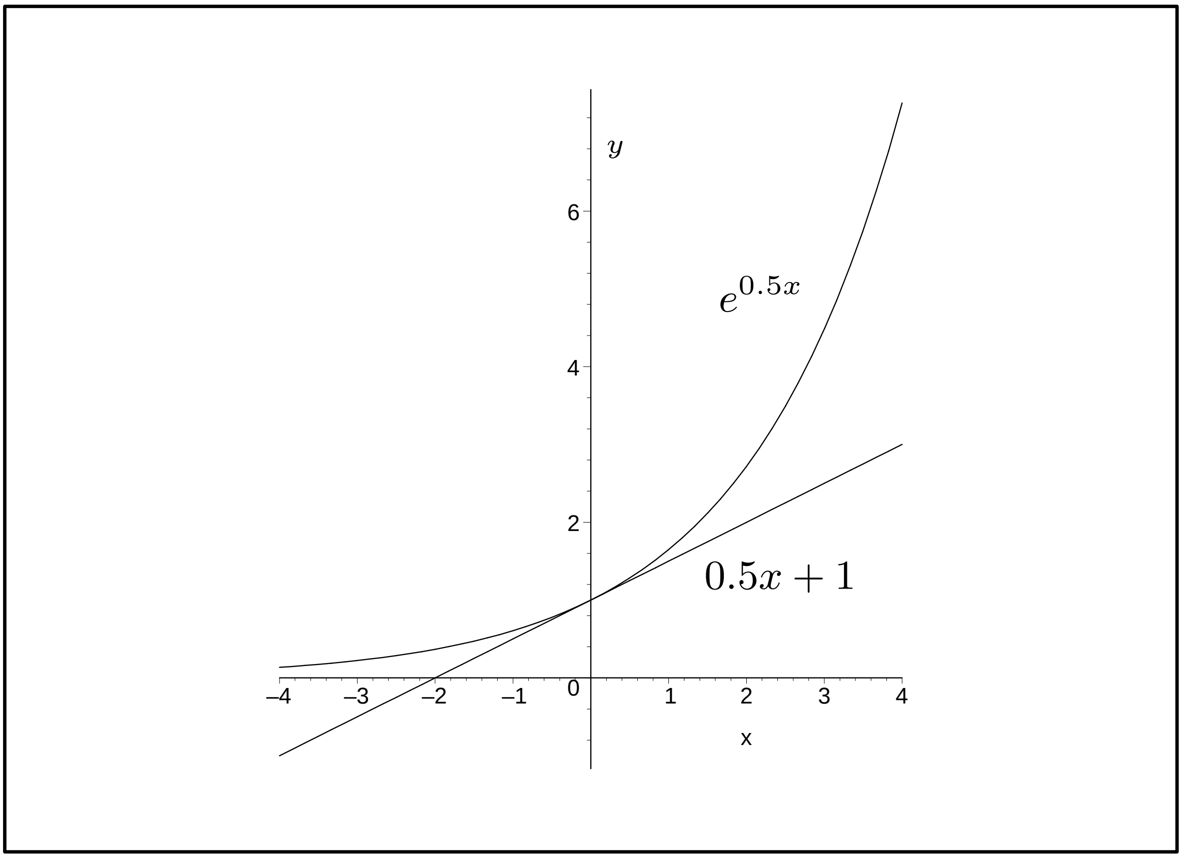
Answer: Remembering that the derivative of e^{kx} is ke^{kx}, we have $y' = 0.5e^{0.5x}$.

At the point $(0, 1)$ the slope of the curve is $y'(0) = 0.5e^0 = 0.5$. Hence the tangent line has equation $y = 0.5x + c$.

To find the value for c, we substitute $(0, 1)$ into this equation, giving $1 = 0.5 \times 0 + c$, so $c = 1$.

Hence the tangent line has equation $y = 0.5x + 1$.

We can check that this is plausible. The figure shows the graph of $y = e^{0.5x}$ and the graph of $y = 0.5x + 1$. Clearly, the tangent line looks to be correct.

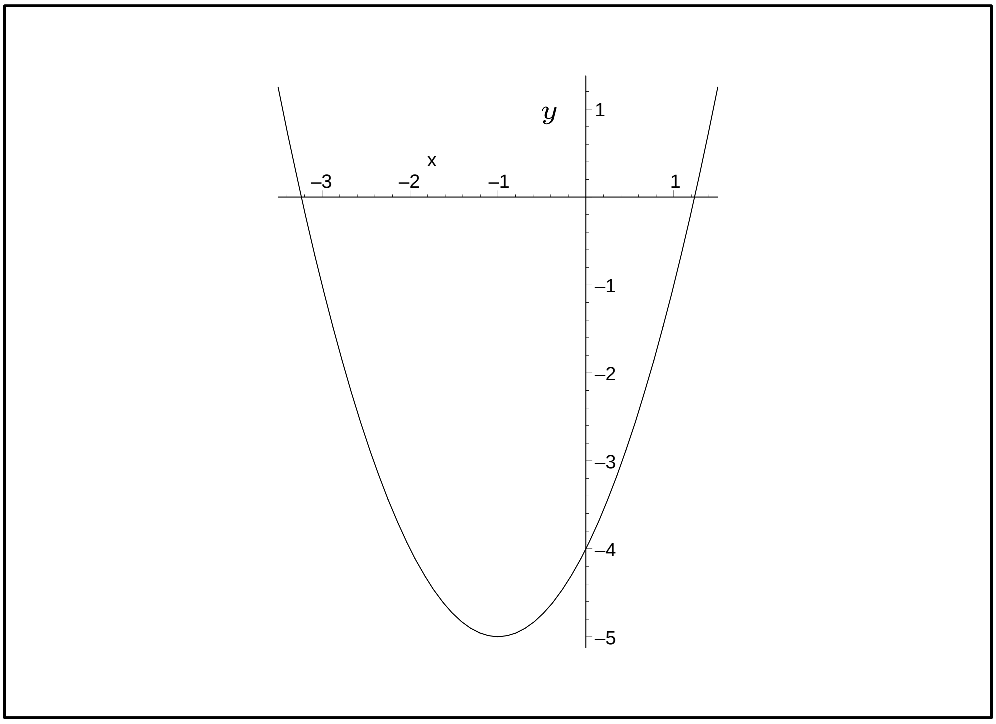


***Question** 12.1.2* Let $f(x) = x^2 + 2x - 4$.

(a) Find the equation of the tangent line to $f(x)$ at $(1, -1)$.

(b) Find the equation of the tangent line at $(-1, -5)$.

(c) Sketch the tangent lines on the graph below.



12.2 Derivatives and motion

- An important use of derivatives is in the study of motion.
- Given a moving object, three quantities with which you should be familiar are:
  - the **displacement** (similar to the **distance**) that the object has travelled;
  - the **velocity** (or **speed**) of the object; and
  - the **acceleration** of the object.
- These words have meanings that are pretty similar to how they are used in everyday conversation.
- One slight difference when studying them mathematically is that there is usually a **direction** associated with them.
- The only directions we will encounter are positive and negative.
- For example, when a ball is thrown in the air, we will assume that the positive direction is upwards, and the negative direction is downwards. Hence:
  - the displacement of the ball is how far it is from the zero position (possibly different to how far it has travelled);
  - a displacement of 0 means the ball is on the ground;
  - a positive displacement means the ball is above the ground;
  - a positive velocity means the ball is travelling upwards;
  - a negative velocity means the ball is travelling downwards; and
  - the acceleration due to gravity is negative, as it is pulling the ball in a downwards direction.

- We are used to talking about distance, speed and acceleration in everyday life. When doing so, we usually include units in the measurements.

***Example** 12.2.1* Some examples of displacement are:

- a 100 metre (100m) race in the Olympic Games;
- an 8,000 kilometre (8000km) plane flight; and
- driving a car for 80 kilometres (80km).

Some examples of velocity are:

- a speed limit of 60 kilometres per hour (60 km/hr);
- a snail moving at 15 millimetres per minute (15 mm/min); and
- a runner travelling at 10 metres per second (10 m/sec).

The units for measuring acceleration are less familiar. Some examples are:

- a rocket accelerating at 5 metres per second, per second (5 m/sec/sec); and
- the acceleration due to gravity on earth of 9.8 metres per second per second (9.8 m/sec/sec).

- In our work we are only concerned with the mathematics, so will usually ignore the units.
- We'll simply describe displacement, velocity and acceleration as numbers that may be positive or negative.
- It will usually be obvious which direction is positive and which is negative.

- In Example 12.2.1, an example of displacement was 80km, and an example of velocity was 60 km/hr.
- Clearly, velocity is a change in distance over a given time period (here, distance changes by 60km as time changes by 1 hour).
- Similarly, acceleration is a change in velocity over a given time period.
- Recall that a derivative is a change in y compared to a change in x.
- Then we have the following important results.

Derivatives and motion.

If displacement, velocity and acceleration are considered as functions of time, then:

- *the derivative of displacement is velocity; and*
- *the derivative of velocity is acceleration.*

- Of course, because velocity is the derivative of displacement and acceleration is the derivative of velocity, acceleration is the second derivative of displacement.
- It is customary to write S for displacement, v for velocity and a for acceleration.
- Thus at any time t, displacement is $S(t)$, velocity is $v(t)$ and acceleration is $a(t)$.
- We have:

$$v = S' \left(= \frac{dS}{dt}\right), \qquad a = v' \left(= \frac{dv}{dt}\right) = S'' \left(= \frac{d^2S}{dt^2}\right)$$

***Example** 12.2.2* A ball is thrown vertically upwards into the air from the ground. It is found that the displacement of the ball above the ground at any time t is given by

$$S(t) = 25t - 5t^2.$$

1. What is the velocity of the ball at any time t?
2. What is the acceleration of the ball at any time t?
3. What is the highest point the ball reaches?
4. When does the ball reach the ground again?

Before answering these questions, it's useful to think about **how** you could answer them.

For (1), all we need to do is differentiate $S(t)$ to get $v(t)$.

For (2), all we need to do is differentiate $v(t)$ to get $a(t)$.

For (3), we need to work out what determines the highest point that the ball reaches. At this point, the ball has just stopped moving up, and is just about to start moving down. Hence its velocity must be 0. So if we solve $v(t) = 0$, we should get the time at which it is at its highest point, and substituting that value of t into $S(t)$ will give us the displacement.

For (4), when it reaches the ground again, the displacement must be 0. Hence if we solve $S(t) = 0$, that will give the time at which the ball reaches the ground again.

continued...

Example *12.2.2 (continued)*

Answer to 1:

The velocity is given by $v(t) = S'(t) = 25 - 10t$.

Answer to 2:

Acceleration is given by $a(t) = v'(t) = -10$.

The negative value for acceleration here means that the acceleration is not in the direction in which S is measured (upwards) but in the opposite direction, back to the ground; of course, a is caused by gravity.

Answer to 3:

We have $v(t) = 0$, so $25 - 10t = 0$,
so $25 = 10t$, so $t = 2.5$. Then we substitute $t = 2.5$ into S, giving:

$$S(2.5) = 25\times2.5 - 5\times2.5^2 = 31.25.$$

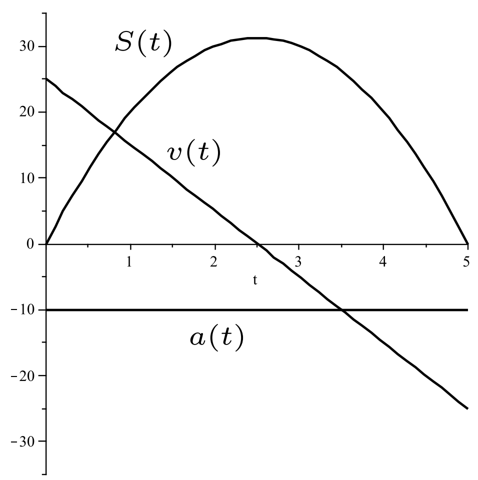
Thus 2.5 seconds after the ball is throw it has zero velocity, and reaches its highest point above the ground of 31.25.

Answer to 4:

Solve $S(t) = 0$. Then $25t - 5t^2 = 0$, which we can solve using the quadratic formula or by factorising. Let's factorise.

We have $t(25 - 5t) = 0$, so $t = 0$ or $25 - 5t = 0$, so $t = 0$ or $t = 5$.

Thus the ball is at ground level at time $t = 0$ (when it is initially thrown), and again at time $t = 5$.



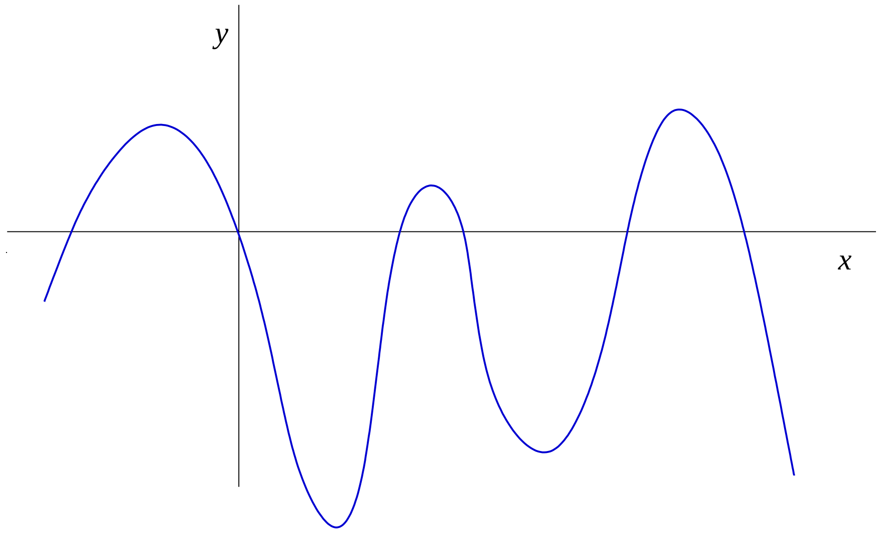
12.3 Local maxima and minima

- A function's derivative gives its slope at any point.
- If the slope is positive then the function is getting larger, or **increasing**.
- If the slope is negative then the function is getting smaller, or **decreasing**.
- Think about the special case where the derivative is 0.
- At such points the function is **neither** increasing nor decreasing.

Critical points.

Given a function f, the **critical points** *or* **stationary points** *of f are those points at which the derivative of f equals 0.*

Question *12.3.1* On the following diagram, identify all regions in which the function is increasing and those in which it is decreasing. Show all critical points.



- Critical points are very important, so we need a technique for finding them.

Finding critical points.

Given a function $f(x)$, to find the critical points of f we

1. *Differentiate f.*
2. *Find any values of x for which $f(x)' = 0$.*
3. *Substitute those values of x into $f(x)$ to calculate the corresponding y values.*

Example *12.3.2* Find all critical points of $f(x) = 2x^2 + 4x + 6$. **Answer**: We have $f'(x) = 4x + 4$.

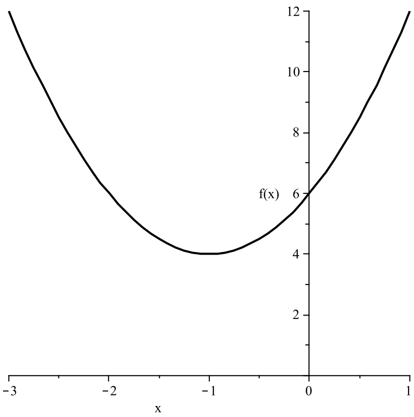
At any critical point we have $f'(x) = 0$.

Hence $4x + 4 = 0$, so $4x = -4$, so $x = -1$.

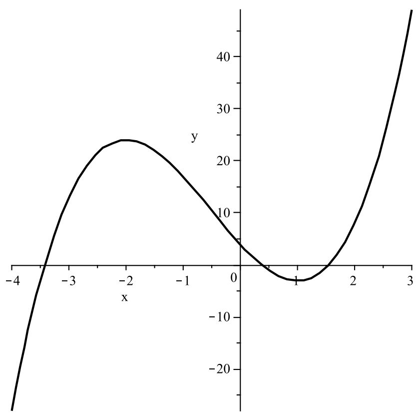
To find the y-value substitute $x = -1$ into the **original** function.

So $f(-1) = 2\times(-1)^2 + 4\times -1 + 6 = 4$.

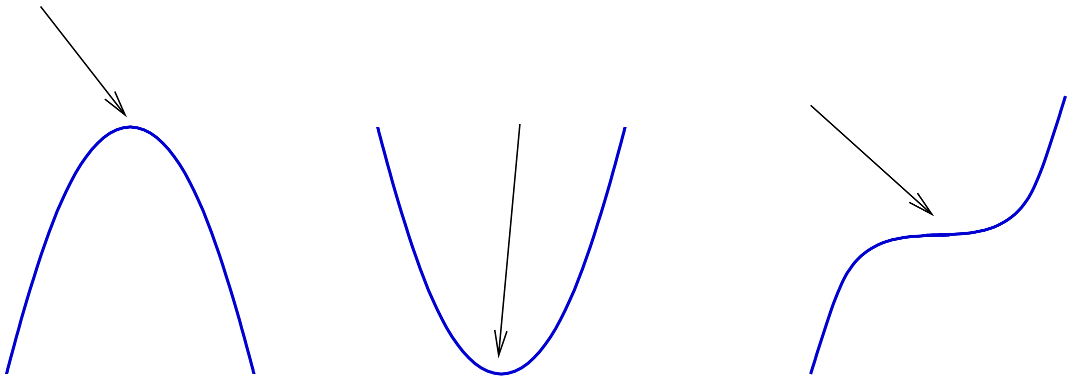
Hence there is one critical point at $(-1, 4)$.



Question *12.3.3* Find all critical points of:
$y = 2x^3 + 3x^2 - 12x + 4$.



- At a critical point we know that the function has slope zero, so is neither increasing nor decreasing.
- The following diagram shows the three ways in which this can happen; in each case, an arrow highlights the critical point.



- The critical point on the left is called a **local maximum**.
- The critical point in the middle is called a **local minimum**.
- The critical point on the right is called a **point of inflection**.
- We will only encounter the first two, so you can ignore points of inflection.
- Be familiar with this terminology:
  - A local maximum is a peak (or hill) and a local minimum is a trough (or valley).
  - The plural of maximum is **maxima** and the plural of minimum is **minima**.
- Usually, rather than simply finding a critical point, you'll be asked to *find and classify* all critical points.
- This means that you need to find all critical points, and then classify each one as a local maximum or a local minimum.

- We'll look at two rules for deciding whether a critical point is a maximum or minimum.
- The first rule involves the first derivative, and thinking a bit about what a maximum or minimum looks like.
- The second rule involves using the second derivative.
- You can use whichever rule you like, but we encourage you to use the second rule as it involves less scope for error.

First derivative test.

Given a function f, there is a **local maximum** *at $x = a$ if:*

$$\begin{aligned} f'(x) &> 0 \text{ for } x < a \text{ (for } x \text{ close to } a) \\ f'(a) &= 0 \\ f'(x) &< 0 \text{ for } x > a \text{ (for } x \text{ close to } a) \end{aligned}$$

There is a **local minimum** *at $x = a$ if:*

$$\begin{aligned} f'(x) &< 0 \text{ for } x < a \text{ (for } x \text{ close to } a) \\ f'(a) &= 0 \\ f'(x) &> 0 \text{ for } x > a \text{ (for } x \text{ close to } a) \end{aligned}$$

- This rule may look complicated, but it makes sense.
- It says that f has a local maximum if f' is positive before $x = a$, zero at $x = a$ and negative after $x = a$; thus the function is increasing, then flat, then decreasing.
- Similarly, it says that f has a local minimum if the function is decreasing, then flat, then increasing.

Second derivative test.

To find all local maxima and minima of $y = f(x)$:

(1) *find the derivative $f'(x)$;*

(2) *find all values of x for which derivative is 0;*

(3) *find the second derivative $f''(x)$;*

(4) *for each value of x at which $f' = 0$, evaluate f'' at that point. If the second derivative value is*

- *positive, then the function has a local minimum at that point;*
- *negative, then the function has a local maximum at that point;*
- *zero, then the test fails and we can't conclude anything. (We won't deal with this situation in MATH1040.)*

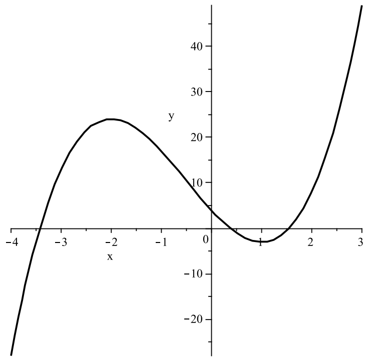
***Example** 12.3.4* (From Example 12.3.2, $f(x) = 2x^2 + 4x + 6$ has a critical point at $(-1, 4)$. Classify this critical point.

We answer this in two ways; using the first derivative test, then the second derivative test.

First derivative test: We have $f'(x) = 4x + 4$, which is zero at $x = -1$. When x is a bit smaller than -1 (say $x = -1.1$), f' is negative. When x is a bit larger than -1 (say $x = -0.9$), f' is positive. Hence the derivative goes negative, then 0, then positive, so the critical point is a local minimum.

Second derivative test: We have $f' = 4x + 4$, so $f'' = 4$. Hence the second derivative is positive, so the critical point truly is a local minimum.

Question *12.3.5* (See Question 12.3.3.) Classify the critical points of $y = 2x^3 + 3x^2 - 12x + 4$.



Question *12.3.6* Find and classify all critical points of the function $y = x^2 - 2x + 4$, then roughly sketch its graph.

12.4 Some practical problems

- Derivatives are useful in solving problems from construction, economics, business operations and product design.
- Most businesses want to maximise profits or minimise costs. If you know the profit or cost functions, then optimal levels of production will occur at the critical points of those functions.

***Example** 12.4.1* The ordering department for a company finds that if they place x orders for materials each year, the total associated costs are $C(x) = 1000x + \dfrac{25000}{x}$ dollars for $x > 0$. What number of orders per year minimises costs, and what do the costs equal at that number of orders?

Answer: To minimise $C(x)$ we first differentiate, giving:

$$C' = 1000 - \frac{25000}{x^2}.$$

Next we let the derivative equal 0. So:

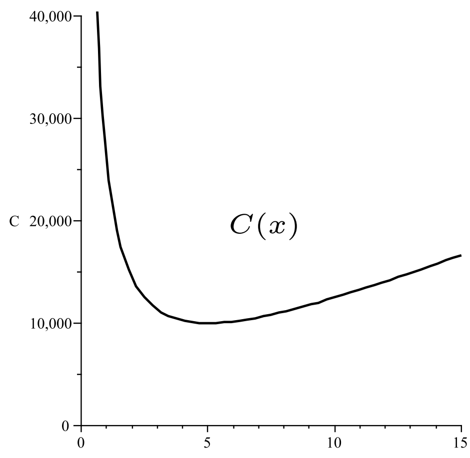
$$\begin{array}{lrcl} & 1000 - \frac{25000}{x^2} & = & 0 \\ \Rightarrow & 1000 & = & \frac{25000}{x^2} \\ \Rightarrow & 1000x^2 & = & 25000 \\ \Rightarrow & x^2 & = & 25 \\ \text{so :} & x = 5 & \text{or} & x = -5. \end{array}$$

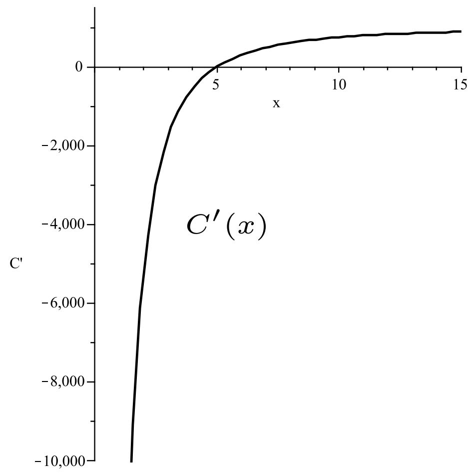
As $x > 0$, so $x = 5$. Next we need to check whether this is a minimum or a maximum. The second derivative is:

$$C'' = \frac{50000}{x^3},$$

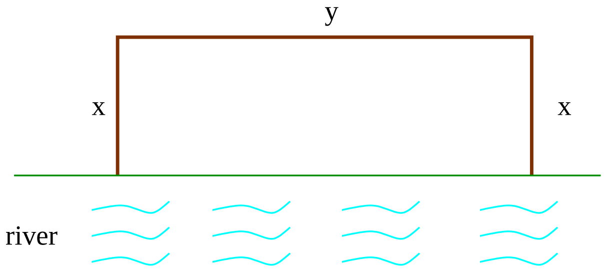
which is positive when $x = 5$. Hence this is a local minimum. Thus costs are minimised when $x = 5$, and the costs equal

$$C(5) = 1000\times 5 + \frac{25000}{5} = \$10000.$$





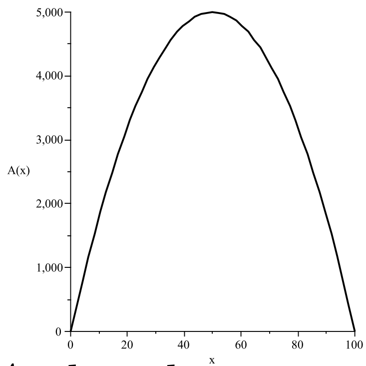
Example *12.4.2* A farmer wants to build a rectangular pen for his sheep. One side is a straight river. For the other three sides he has 200m of fencing to use. What is the maximum area of pen he can make?



Let the pen be as shown, so the fenced perimeter is of length $2x + y$. If all 200m of material is used then $2x + y = 200$, so $y = 200 - 2x$. The area of the pen is $A = xy$.

Before we solve the problem, let's see that it's sensible to use a maximisation approach. We'll try various values of x and y, and see what happens to the area.

When $x = 1$, $y = 198$ and $A = xy = 198$.
When $x = 2$, $y = 196$ and $A = 2\times196 = 392$.
When $x = 3$, $y = 194$ and $A = 3\times194 = 582$.
When $x = 10$, $y = 180$ and $A = 1800$.
When $x = 95$, $y = 10$ and $A = 950$.



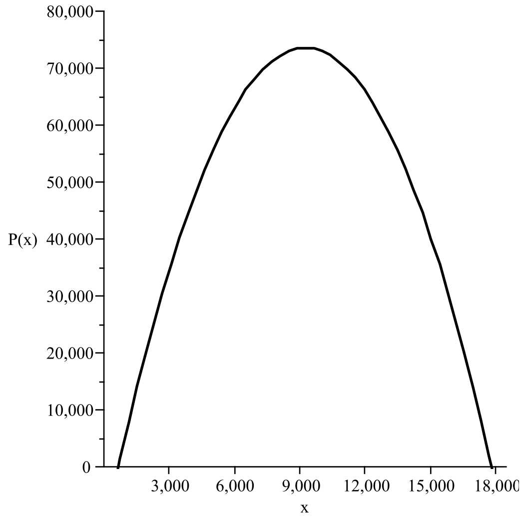
Hence, as the values of x and y change, A also changes, so it seems sensible that some values for x and y maximise A.

Substituting $y = 200 - 2x$ into the expression for A, gives $A = x(200 - 2x) = 200x - 2x^2$.

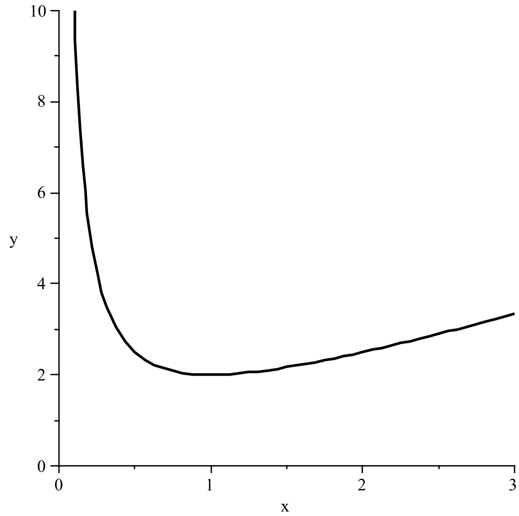
So $A' = 200 - 4x$, and this equals 0 when $x = 50$. Thus we have a critical point when $x = 50$. Now $A'' = -4$, which is negative, so the critical point is a local maximum.

Hence when $x = 50$, the area A is a maximum, equal to $50(200 - 100) = 5000\text{m}^2$, and the pen measures $50\text{m} \times 100\text{m}$.

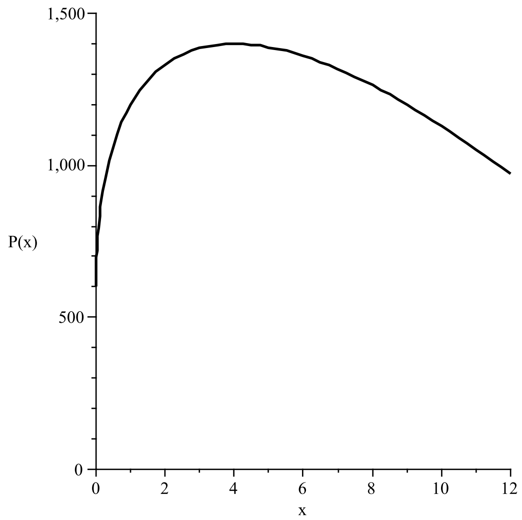
Question *12.4.3* The cost of publishing a book is \$12,000 plus \$6.50 per copy. The demand function is $p(x) = 25 - 0.001x$, which gives the price \$$p$ which should be charged in order to sell x copies. If profits are to be maximised, how many should be produced, and what should the selling price be?



Question *12.4.4* Find the smallest possible value of the sum of a positive real number x and its inverse.



Question *12.4.5* The base profit per hectare of a certain crop is \$600. Each tonne of fertiliser costs \$200, and the additional profit from applying x tonnes of fertiliser per hectare is $800\sqrt{x}$. What amount of fertiliser per hectare will maximise profit?



NOTES

NOTES

13 Integration

Why are we covering this material?

- We have seen that differentiation allows us to find the derivative or slope of the function; this has many uses.
- Sometimes, the reverse step is useful: given the derivative of a function, how can we find the original function?
- The reverse process is called **integration**.
- Just as the derivative of a function gives the slope of the function, there is also a useful geometric interpretation of the integral: the integral of a function gives the area between the function and the x-axis.
- We don't cover integration in much detail; if you do more calculus, you'll study it in much more detail.
- We'll see a few rules which make the task easier, but mostly we'll use *smart trial and error*.
- **Topics in this section are**
  - Introduction to integration.
  - Rules for integration.
  - Initial conditions.
  - Definite integrals and areas.
  - Integrals and motion.

13.1 Introduction to integration

- Given a function, we can usually find its derivative.
- We have seen a number of rules for finding derivatives.
- What about the reverse step? That is, if we are given the derivative of a function, can we find the original function?

Example *13.1.1* Let $F(x)$ be an unknown function whose derivative is $6x$. Find $F(x)$.

Answer: Use smart trial and error, and think about how differentiation works. When we differentiate x to some power, we subtract one from the power. Here, the derivative includes x to the power of 1, thus the original function must have involved x^2. So let's try x^2 as our first guess at the original function. Differentiate x^2 and we get $2x$, but we wanted to get $6x$. Hence the original function should be 3 times what we guessed, so it is $3x^2$. (Check! The derivative of $3x^2$ is $6x$.)

Question *13.1.2* Find $F(x)$ where $F(x)$ is an unknown function whose derivative is $3x^2$.

Find $G(x)$ where $G(x)$ is an unknown function whose derivative is e^x.

Find $H(x)$ where $H(x)$ is an unknown function whose derivative is $4x^3 + 3x^2 + 2x + 1$.

Integration.

The process of finding an unknown function $F(x)$ from its derivative $f(x)$ is called **integration**.

We say that $F(x)$ is an **antiderivative** *or* **integral** *of $f(x)$.*

We write $F(x) = \int f(x)\, dx$, pronounced "F(x) is the integral of $f(x)$ with respect to x".

$\int f(x)\, dx$ is called an **indefinite integral**.

- Don't be confused by the notation in integration.
- The $\int$ sign simply means 'the integral of'.
- The dx alludes to the dx we saw in $\dfrac{dy}{dx}$.
- Integration is much harder to do than differentiation.
- In this course we will only cover some fairly basic functions, whose integrals are fairly easy to find.
- The technique we will use can be summarised as follows.
  - To find $F(x) = \int f(x)\, dx$, make an informed guess as to what $F(x)$ might be.
  - Differentiate $F(x)$ and see if you get the right answer.
  - If you are not correct, change your guess for $F(x)$ and repeat.
- The following example illustrates an important feature of integration: multiple functions have the same derivative.

Example *13.1.3* Let $F(x) = x^2 + 4$, $G(x) = x^2 - 2$ and $H(x) = x^2$.

Then $F'(x) = 2x$, $G'(x) = 2x$ **and** $H'(x) = 2x$.

Then what does $\int 2x\, dx$ equal?

Does it equal $F(x)$, or $G(x)$, or $H(x)$, or even something else?

- When we differentiate, the constant term disappears.
- Thus, when we integrate, we don't know what the constant term was.
- The constant could have been any value at all.
- To resolve this we take a special step.

Constant of integration.

Whenever we find an indefinite integral, we include a constant of integration *in the answer, which takes the form of '*$+C$*'.*

Example *13.1.4*

$$\int 2x\, dx = x^2 + C \qquad \int 3x^2\, dx = x^3 + C$$

$$\int 7\, dx = 7x + C$$

Question *13.1.5* Find $\int (3x^2 + 2x + 4)dx$.

13.2 Rules for integration

- We said before that the main technique we shall use is "guess the answer and check it".
- There are a few rules which help us to improve our guesses.
- Look at the following rules: each comes from a rule for differentiation.
- There are examples on the next page.

Rules for integration.

Rule 1

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$

Rule 2 *If k is a non-zero constant, then:*

$$\int k f(x) \, dx = k \int f(x) \, dx.$$

Rule 3

$$\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx.$$

Rule 4 *If k is a non-zero constant then:*

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C.$$

Rule 5

$$\int \cos x \, dx = \sin x + C, \qquad \int \sin x \, dx = -\cos x + C.$$

Rule 6

$$\int \frac{1}{x} \, dx = \ln x + C.$$

Example *13.2.1* Here are some examples of Rule 1:

$$\int x^2\,dx = \frac{1}{2+1}x^{2+1} + C = \frac{1}{3}x^3 + C.$$

$$\int x^4\,dx = \frac{1}{4+1}x^{4+1} + C = \frac{1}{5}x^5 + C.$$

Example *13.2.2* Here are some examples of Rule 2:

$$\int 3x^2\,dx = 3\times\int x^2\,dx = 3\times\frac{1}{3}x^3 + C = x^3 + C.$$

$$\int 10x^4\,dx = 10\times\int x^4\,dx = 10\times\frac{1}{5}x^5 + C = 2x^5 + C.$$

Example *13.2.3* Here are some examples of Rule 3:

$$\int 3x^2 + 2x\,dx = \int 3x^2\,dx + \int 2x\,dx = x^3 + x^2 + C.$$

$$\int e^x + 1\,dx = \int e^x\,dx + \int 1\,dx = e^x + x + C.$$

Example *13.2.4* Here are some examples of Rule 4:

$$\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C.$$

$$\int 6e^{2x}\,dx = 3e^{2x} + C.$$

Example *13.2.5* Here are some examples of Rule 5:

$$\int \sin x - 2\,dx = -\cos x - 2x + C.$$

$$\int 2x + \cos x\,dx = x^2 + \sin x + C.$$

Example *13.2.6* Here are some examples of Rule 6:

$$\int \frac{1}{x} + 2x\,dx = \ln x + x^2 + C.$$

$$\int \frac{2}{x}\,dx = 2\ln x + C.$$

- Any problem you encounter will be solved by using the six rules in conjunction with guess and check.

Question *13.2.7* Find $\int 4x - 6x^2\,dx$.

Question *13.2.8* Find $\int e^{3x} + 4\,dx$.

Question *13.2.9* Find $\int \frac{1}{x} + 2x + 3x^2\,dx$.

***Question** 13.2.10* Find $\int 0 \, dx$.

***Question** 13.2.11* Find $\int 4 + 6t \, dt$.

***Question** 13.2.12* Find $\int 3 \sin x + 2 \cos x \, dx$.

13.3 Initial conditions

- When we covered exponentials, we briefly discussed *initial conditions*.
- For example, we said things like 'the population at time $t = 0$ is 100'.
- This gave us extra information about the problem being studied.
- Initial conditions are also useful when integrating, as they allow us to find an exact value for the constant of integration.

***Example** 13.3.1* Given $F(x) = \int (2x+4)\, dx$ and $F(1) = 0$, find $F(x)$.

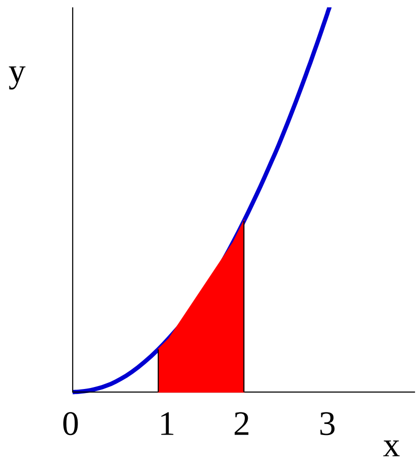
Answer: Clearly, $F(x) = x^2 + 4x + C$. But $F(1) = 0$, so $1^2 + 4 \times 1 + C = 0$, so $C = -5$. Hence $F(x) = x^2 + 4x - 5$.

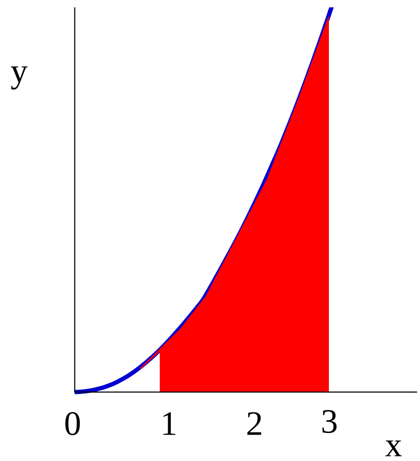
Note that $F(x) = x^2 + 4x - 5$ is the **only** function whose derivative is $2x + 4$ and which has $F(1) = 0$.

***Question** 13.3.2* Suppose $F(x) = \int e^{2x}\, dx$, and $F(0) = 1$. Find $F(x)$.

13.4 Definite integrals and areas

- Until now, we have simply said that indefinite integrals are the opposite of derivatives.
- There is another very important interpretation of integrals.
- In fact, the integral of a function gives us the area under the graph of the function; that is, between the graph and the x-axis.
- Showing that this is true (and also considering what happens when the answer is negative) is beyond the scope of this course.
- The following two graphs each show the graph of $y = x^2$ as x goes from 0 to 3.
- On the left we have shaded the area under the curve from $x = 1$ to $x = 2$.
- On the right we have shaded the area from $x = 1$ to $x = 3$.
- Clearly, the two areas are different.





- We need some way to specify how much of the area we want to find: clearly, the area from $x = 1$ to $x = 2$ is smaller than the area from $x = 1$ to $x = 3$.

- To do this, we need to introduce a new type of integral, called the **definite integral**.
- The definite integral looks very similar to the indefinite integral which we have already covered.
- The main difference is that we now include **two limits of integration**, and we do something extra after we have found the antiderivative.
- The limits of integration are the values of a and b in the following definition.
- They specify the range of x values to use for the area calculation; in the definition, we will go from $x = a$ to $x = b$.

Definite integral.

Given a curve $f(x)$, to find the area under the curve from $x = a$ to $x = b$, we write:

$$\int_a^b f(x)\,dx,$$

which is pronounced 'the integral of $f(x)$ from $x = a$ to $x = b$'.

***Example** 13.4.1* We could calculate the areas under the curves on the previous page by finding

$$\int_1^2 x^2\,dx \quad \text{and} \quad \int_1^3 x^2\,dx.$$

- A vital theorem shows us how to use the antiderivative to find this area under the curve.

Fundamental theorem of calculus.

If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

- We can restate the fundamental theorem of calculus as a simple procedure.

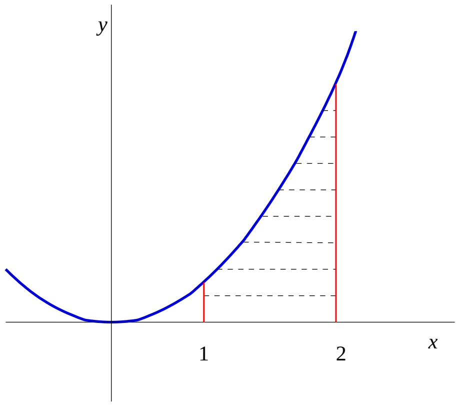
Finding definite integrals.

To evaluate $\int_a^b f(x)\,dx$:

1. *Find an antiderivative of $f(x)$, say $F(x)$.*
2. *Evaluate $F(b)$.*
3. *Evaluate $F(a)$.*
4. *Calculate $F(b) - F(a)$.*

- There is a standard way for setting out these problems.
- First, find the antiderivative.
- Then, surround this expression by large square brackets.
- Write the limits of integration outside the brackets, the larger value at the top and the other value at the bottom.
- Evaluate the antiderivative at the top limit of integration.
- Evaluate the antiderivative at the bottom limit of integration.
- Subtract the two to obtain the answer.
- Study the following example.

***Example** 13.4.2* Find the area under the curve $y = x^2$ from $x = 1$ to $x = 2$.



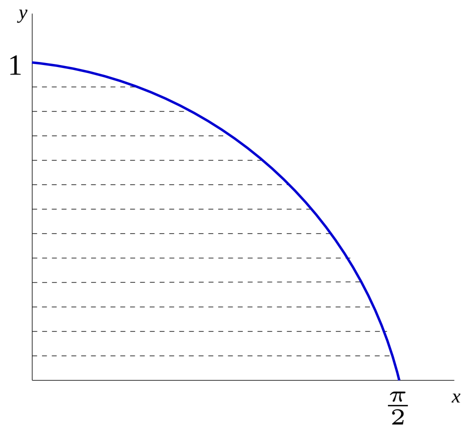
The area is equal to $\int_1^2 x^2 \, dx$. Then:

$$\begin{aligned}
\int_1^2 x^2 \, dx &= \left[\frac{1}{3}x^3 + C\right]_1^2 \\
&= \left(\frac{1}{3}\times 2^3 + C\right) - \left(\frac{1}{3}\times 1^3 + C\right) \\
&= \left(\frac{8}{3} + C\right) - \left(\frac{1}{3} + C\right) \\
&= \frac{8}{3} + C - \frac{1}{3} - C = \frac{7}{3}.
\end{aligned}$$

***Question** 13.4.3* Find $\int_0^2 2x \, dx$.

- Notice what happens to the constant of integration when calculating definite integrals: it **always** cancels out.
- When calculating definite integrals, you don't need the constant of integration. You can include it if you like, but you must remember to cancel it before giving your final answer.

***Example** 13.4.4* Find the area under the curve $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$.



Area is $\displaystyle\int_0^{\frac{\pi}{2}} \cos x dx = \Big[\sin x\Big]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1.$

***Question** 13.4.5* Find $\displaystyle\int_1^2 (3x^2 + 2x + 1)dx$.

Question *13.4.6* Find $\displaystyle\int_0^1 (2x + e^x)\,dx$.

Question *13.4.7* Find $\displaystyle\int_1^e (x^{-1} + 2x)\,dx$.

13.5 Integrals and motion

- We have seen that the derivative of displacement is velocity, and the derivative of velocity is acceleration.
- We also know that the reverse of differentiation is integration.
- Combining these concepts, we can see that **velocity** is the **integral of acceleration**, and **displacement** is the **integral of velocity**.
- Thus, given an expression for an object's acceleration we can work out its velocity at any time, and we can work out its displacement at any time.
- Usually in these problems we use t (time) instead of x to represent the independent variable, so we will integrate with respect to t.
- Often there will be an initial condition, such as "displacement at time $t = 0$ is 0". We can use this initial condition to obtain an exact value for the constant of integration.

***Example** 13.5.1* A non-moving car with displacement 0 accelerates at a constant rate of $a = 2$. Find expressions for its velocity $v(t)$ and displacement $S(t)$ at any time t.

We have $v(t) = \int a(t)\, dt = \int 2\, dt = 2t + C$.
At $t = 0$ the car is stopped, so $v(0) = 0$, so $C = 0$. Hence $v(t) = 2t$.
Now $S(t) = \int v(t)\ dt = \int 2t\ dt = t^2 + C$. At $t = 0$ the displacement is 0, so $S(0) = 0$, so $C = 0$. Hence $S(t) = t^2$.

Question *13.5.2* A rocket takes off vertically from a launch pad at time $t = 0$, with velocity $v(t) = 2t+1$ metres per second. At $t = 1$, the displacement of the rocket is 6 metres.

(a) Find an expression for the rocket's displacement $S(t)$ at any time t.

(b) When does the rocket's displacement equal 34 metres?

***Question** 13.5.3* Using the information from Question 13.5.2, find how far the rocket travels between time $t = 3$ and $t = 8$. Find this answer in 2 ways, by:

(a) substituting values for t into the expression for displacement found in Question 13.5.2; and

(b) solving the problem as a definite integral.

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