THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

MATH1040

Basic Mathematics

Revision Guide

1st Edition, 2010

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1 Numbers and arithmetic

This material is fundamental to all the maths you will do in MATH1040, and in many courses you'll take at University. It's even useful in everyday life! Some of you will be quite familiar with this material, but many others will not. If you can't do this stuff, you will be unable to do the harder and more interesting things later in the course.

Topics in this section are

- Thinking about Maths.
- Types of numbers.
- Number lines and order.
- Absolute values.
- Simple mathematical operations.
- Order of operations.
- Introduction to exponentiation.
- Square roots.
- Prime numbers and factors.
- Fractions.

1.1 Thinking about Maths

Mathematics is often not easy, but it's important! You will encounter some sort of maths almost every day of your life. Most people need to think quite hard when doing maths, but there are some skills and tricks that really help. Two important approaches you must learn to use are **estimating** and **checking your answers**.

Most questions in maths have one correct final answer, but many different ways of getting that answer. Usually, you can use any valid method you like to get the answer, although sometimes you may be asked to use a particular method. It's important that you show the steps you take, as many marks will usually be allocated to your working.

Often before (or instead of) solving a maths problem exactly, it's useful to quickly **estimate** a rough answer. You can do this by approximating some of the numbers, thus simplifying the calculations. Your answer will not be exactly correct, but it should be "close" to the real answer. The context of the question will determine how accurate you need to be. Sometimes it's good enough to be quite rough.

> ***Example*** *1.1.1* Peter works 36.25 hours per week, and earns \$32.6174 per hour. Estimate his weekly income.
>
> **Answer:** The key word here is estimate. We just need a rough figure, so we can round Peter's working hours to 40 hours per week, and his hourly wage to \$30 per hour, giving us an easy calculation we can do in our head: $40 \times 30 = \$1200$.
>
> For reference, the exact answer is \$1182.38. Note that we rounded the number of hours up, and the hourly pay-rate was rounded down; this helped to increase the accuracy of the estimate.

Be careful how you use estimation. In most cases, problems need **exact** answers. Estimation can be used to check whether or not your final answer is reasonable, but shouldn't be used to find the actual answer.

It's easy to make mistakes when answering a question. Whenever possible, you should **check your answer**. There are many ways of doing this. The list below outlines a few strategies that you can use to check your answers.

- Where appropriate, ask yourself: 'Does the answer make sense in a real-life context'?
- Use estimation to check whether the answer is 'plausible'.
- Check each of the steps in your working.

It is important that you exercise caution while checking solutions. Also, don't readily trust your calculator - it does exactly what you tell it to do and it is very easy to input something incorrectly.

It is also very important that you are able to interpret and understand all of the different mathematical symbols and notations used within the course. Taking the time to develop a thorough understanding of notation will be essential in comprehending the more complex material that arises as the course progresses.

Questions

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1. For each question, decide which answer(s) are most likely to be correct. Explain why.

(**1**) In an Olympic 400m running race, the maximum speed attained by the winner (in metres per second) is:

i. 63.7 ii. 10 iii. 2

(**2**) $1000 is invested in a bank account earning 8% interest per annum for 3 years. What is the final balance?

i. $16728.33 ii. $827.67 iii. $1259.71 iv. $1412.68

2. One Australian dollar (AUD) is worth 0.878 United States dollars (USD). Big Bad John spends 10 nights at a hotel in Las Vegas, at 93 USD per night. His credit card company charges a 1.5% fee to convert from USD to AUD. Roughly estimate his bill in AUD.

3. Roughly estimate the number of babies born in Australia each year.

This section might seem easy, but it's so important that we covered it first. For the rest of this course (and others), make sure you use estimation where appropriate, and always check your answers.

1.2 Types of numbers

Numbers are one of the most fundamental concepts in human existence, and underpin almost everything you see and do. It is necessary that we distinguish carefully the various kinds of numbers. The simplest numbers are the "counting numbers"

$$1, 2, 3, \ldots$$

These positive whole numbers are called the **natural numbers**, the symbol for which is $\mathbb{N}$. Because of their simple nature, there are limited applications of the natural numbers. These deficiencies can be partially overcome by extending this system to include the negative whole numbers and zero (0). This collection of numbers is referred to as the **integers**, the symbol for which is $\mathbb{Z}$, and contains all positive and negative whole numbers and zero, but does not include any decimals.

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

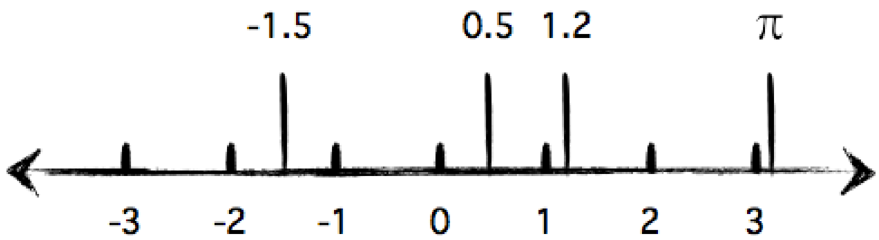
The next extension is obtained by taking quotients, $\frac{m}{n}$, of integers, meaning we have a fraction where both m and n are integers and n is not zero. These numbers are called **rational numbers**, and the set of all rational numbers is denoted by $\mathbb{Q}$. Rational numbers can be written as common fractions or as decimal fractions which either

- **terminate**, for example, $\frac{1}{4} = 0.25$ and $\frac{-7}{5} = \frac{7}{-5} = -\frac{7}{5} = -1.4$ or
- **recur**, for example, $\frac{1}{3} = 0.3333\ldots = 0.\dot{3}$ and $\frac{8}{11} = 0.72727272\ldots = 0.\dot{7}\dot{2}$.

It is important to understand that the rational numbers include the integers. This is because each integer can also be written as a quotient of integers, for example $3 = \frac{3}{1} = \frac{9}{3}$. Note that zero can be written in this form, i.e., $0 = \frac{0}{1} = \frac{0}{-23}$, but dividing by zero is undefined.

There are also numbers that cannot be written as a quotient of integers but can be represented as decimals that neither terminate nor repeat. These numbers are called **irrational numbers**, denoted $\bar{\mathbb{Q}}$, and together with the rational numbers form the set of **real numbers**, denoted $\mathbb{R}$. Some examples of irrational numbers are $\sqrt{2} = 1.4142\ldots$, $\pi = 3.1415\ldots$ The decimal representations of these numbers have no pattern and continue for ever. Every year on Pi Day (March 14 - 3rd month, 14th day), people try to remember as many digits of π as they can. (π is the ratio between a circle's circumference and diameter.) The current world record is 67,890 decimal places, set in 2005 by Chao Lu. He lasted 24 hours and 4 minutes before making a mistake!

1.3 Number lines and order



A number line (sometimes called a *real line*) shows the **order** of the real numbers. Each real number occurs somewhere on the number line. The gaps between the integers on the number line are completely filled by rational and irrational numbers. We can loosely think of the number line as going from $-\infty$ (negative infinity) on the left to ∞ (infinity) on the right.

Number lines are ideal for displaying relationships between numbers. For any two numbers, there are 5 common ways of writing the relationship between these numbers:

$=$: meaning equal to, for example $2 = \frac{6}{3}$;

$<$: meaning less than, for example $1 < \pi$;

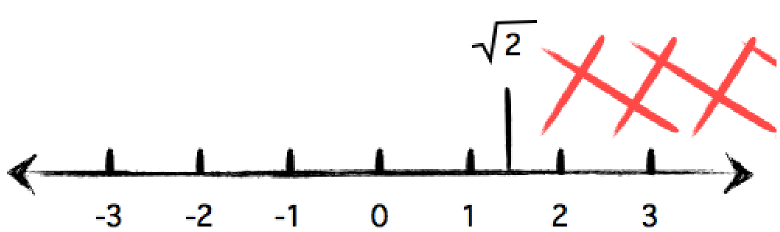
$>$: meaning greater than, for example $0 > -2$;

$\leq$: meaning less than **or** equal to, for example $2 \leq 17$;

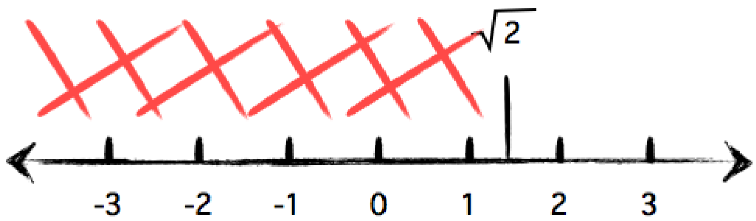
$\geq$: meaning greater than **or** equal to, for example $-1 \geq -1$.

$2 \geq 2$, but people sometimes get confused by this. Think of it this way: if your boss tells you can have at least two days off, this means you can have 2, 3, 4 or more days off. *At least* is the same as *greater than or equal to*.

If we mark a number, say $\sqrt{2}$, an irrational number, on a number line, all numbers that are to the right of $\sqrt{2}$ are 'greater than' or larger than $\sqrt{2}$,



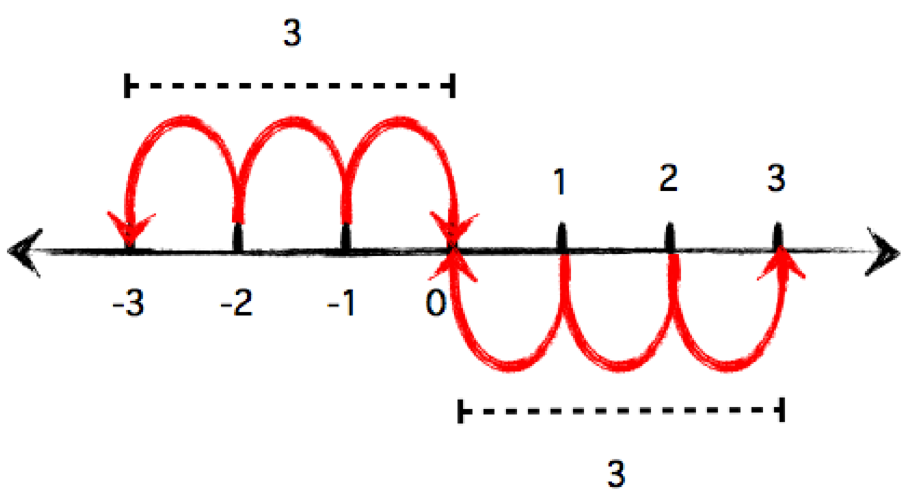
where as all numbers that lie to the left of $\sqrt{2}$ are 'less than' or smaller than $\sqrt{2}$.



1.4 Absolute values

On a number line, all numbers to the left of zero are negative, written with a $-$ sign before them (i.e. -3). There is a special relationship between a number and its negative: both are exactly the same distance from 0 (in opposite directions).

***Example** 1.4.1* -3 and 3 are both a distance of 3 from the point 0.



3 is three steps to the right of 0, -3 is three steps to the left of 0.

Sometimes we are interested in how far the number is from 0, but we don't care in which direction. The **absolute value** of a number is its *distance from zero*. For any number x, we write $|x|$ to represent the absolute value of x. In the example above we found that $|3| = 3$ and $|-3| = 3$. Note that the absolute value of a real number is always either positive or zero, but never negative.

***Example** 1.4.2* Evaluate each of the following:

(a) $|-2|$ **(b)** $\left|\dfrac{7}{2}\right|$ **(c)** $|\,0\,|$ **(d)** $-|\,5\,|$

Solutions:

(a) -2 is a distance of 2 from 0, so $|-2| = 2$.

(b) We can take the absolute value of any real number. $\frac{7}{2}$ is a distance of $\frac{7}{2}$ from 0, so $|\,\frac{7}{2}\,| = \frac{7}{2}$.

(c) 0 is a distance of 0 from 0, so $|\,0\,| = 0$.

(d) We read this as the negative of the absolute value of 5. Since $|\,5\,| = 5$ then $-|\,5\,| = -5$.

We will see more on absolute values later in the course.

Questions

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1. Evaluate each of the following:

(**1**) $|-7.82|$

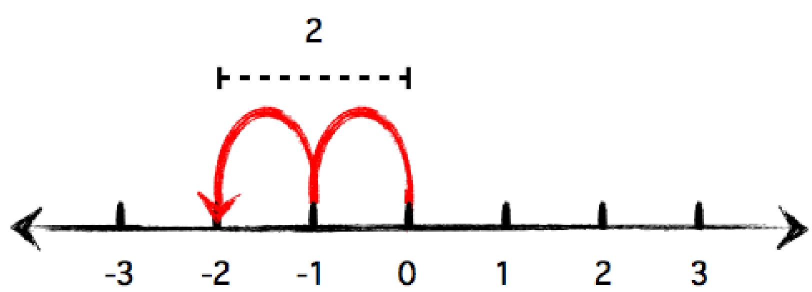
(**2**) $-|-1|$

(**3**) $|-2+5|$

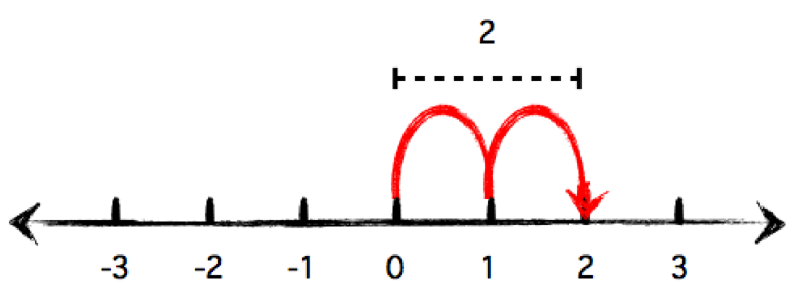
1.5 Simple mathematical operations

You should be quite familiar with addition, multiplication, subtraction and division. What may be unfamiliar is using these operations with mixtures of positive and negative numbers. In the section on absolute values we talked about the relationship between a number and its negative: both being exactly the same distance from 0, in opposite directions. What wasn't explicitly stated in that section was that the negative of a negative number is positive.

***Example** 1.5.1* Show that $-(-2) = 2$ using a number line.



-2 is two steps to the left of 0. This mean that the negative of -2 must be two steps to the right of zero.



2 is two steps to the right of zero and thus $-(-2) = 2$.

We now outline below the rules which govern arithmetic on combinations of positive and negative numbers:

- Adding a negative number is equivalent to subtracting that number. For example, $3 + (-4) = 3 - 4 = -1$. Note that it will often be useful to use this rule to treat subtraction as the addition of the corresponding negative number.

- Subtracting a negative number is equivalent to adding the positive of that number. For example, $2 - (-3) = 2 + (-(-3)) = 2 + 3 = 5$.

- Multiplying or dividing two numbers that are both either positive or negative will result in a positive answer. For example, $(-3) \times (-2) = +(3 \times 2) = 6$.

- Multiplying or dividing two numbers where one is positive and one is negative will result in a negative answer. For example, $(-3) \times 5 = -(3 \times 5) = -15$.

These last 2 points can be summarised in the following table. Here +ve means positive and -ve means negative.

1st number	$\times$ or $\div$	2nd number	answer
$+$ve	$\times$ or $\div$	$+$ve	$+$ve
$+$ve	$\times$ or $\div$	$-$ve	$-$ve
$-$ve	$\times$ or $\div$	$+$ve	$-$ve
$-$ve	$\times$ or $\div$	$-$ve	$+$ve

***Example** 1.5.2* Understand each of the following:

(a) $4 + (-3) = 4 - 3 = 1$

(b) $(-4) + 3 = -1$ People often make mistakes with this type of question.
We are starting at -4 and adding 3, which means moving 3 steps to the right.
Drawing a number line is a good way to understand.

(c) $4 - (-3) = 4 + 3 = 7$

(d) $(-4) - (-3) = (-4) + 3 = -1$

(e) $(-6) \times 2 = -(6 \times 2) = -12$

(f) $(-6) \times (-2) = +(6 \times 2) = 12$

(g) $6 \div (-2) = -(6 \div 2) = -3$

(h) $(-6) \div (-2) = +(6 \div 2) = 3$

Great care needs to be taken when performing arithmetic on expressions involving zero. When dealing with zero remember the following:

- It is **never** possible to divide by zero. Dividing by zero is undefined (including $\frac{0}{0}$).
- Zero divided by any non-zero number equals zero; for example, $0 \div 7 = 0$
- Any number multiplied by zero equals zero; for example, $0 \times (-7) = 0$
- If two numbers multiply to give zero, then at least one of the numbers must equal zero. We will see this last point later on in the course, so please remember it.

Questions

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1. Evaluate each of the following:

(**1**) $2 - (-3)$

(**2**) $(-5) + 7$

(**3**) $(-6) + (-3)$

(**4**) $3 \times (-4)$

(**5**) $(-10) \div (-5)$

1.6 Introduction to exponentiation

It is important that you are familiar with **power** (or **exponent**) form. Exponential notation is essentially shorthand for the repeated multiplication of a number, much in the same way as multiplication is shorthand for the repeated addition of a number. If the number 3 is multiplied by itself 5 times $(3 \times 3 \times 3 \times 3 \times 3)$ we can write this in exponential form as 3^5, where 3 is the **base** and 5 is called the **power** or **exponent** or **index**.

For example, $4 \times 4 = 4^2$ which is read as 4 *squared* or 4 to the power of 2;

$4 \times 4 \times 4 = 4^3$ which is read as 4 *cubed* or 4 to the power of 3.

Once we start dealing with powers greater than 3 we simply use "to the power of" terminology, for example $4 \times 4 \times 4 \times 4 \times 4 = 4^5$ is read as 4 to the power of 5.

We now want to take a closer look at **exponentiation and negative numbers**. Recall from the last section that multiplying two numbers that are both negative will result in a positive answer, and that multiplying two numbers where one is positive and one is negative will result in a negative answer. We need to think about what this means when dealing with exponents of negative numbers.

To investigate this further we look at the powers of -1, noting that 1 to any power is 1 (as no matter how many times we multiply 1 by itself we always get 1).

Example *1.6.1*

$$\begin{aligned}(-1)^2 &= (-1)\times(-1)\\ &= +(1\times 1)\\ &= 1\end{aligned}$$

Notice that when squaring -1 we multiply -1 by itself, as squaring means multiplying by itself, giving an answer of 1.

Example *1.6.2*

$$\begin{aligned}(-1)^3 &= (-1)\times(-1)\times(-1)\\ &= \big(+\,(1\times 1)\big)\;\times\;(-1)\\ &= 1\times(-1)\\ &= -(1\times 1)\\ &= -1\end{aligned}$$

In this example we initially group and then multiply a pair of -1s to give 1, leaving one -1 unpaired. We then finish the remainder of the multiplication giving us an answer of -1.

Example *1.6.3*

$$
\begin{aligned}
(-1)^4 &= (-1)\times(-1)\times(-1)\times(-1)\\
&= \big(+\,(1\times 1)\big)\ \times\ \big(+\,(1\times 1)\big)\\
&= 1\times 1\\
&= 1
\end{aligned}
$$

This time we are able to pair off and then multiply all the -1s. This is because we have an even number of -1s. After we have paired and multiplied the -1s we are simply left with 1×1 giving us an answer of 1.

Example *1.6.4*

$$
\begin{aligned}
(-1)^5 &= (-1)\times(-1)\times(-1)\times(-1)\times(-1)\\
&= \big(+\,(1\times 1)\big)\ \times\ \big(+\,(1\times 1)\big)\ \times(-1)\\
&= 1\times 1\times(-1)\\
&= 1\times(-1)\\
&= -\,(1\times 1)\\
&= -1
\end{aligned}
$$

In this example we are able to pair off and then multiply all but one of the -1s. This is because we have an odd number of -1s. In the final step of the multiplication we are left with 1×-1, this results in an answer of -1.

In the above examples we noticed that when the exponent is **even**, the answer is **positive**; and when the exponent is **odd**, the answer is **negative**. This is true for the exponentiation of all negative numbers, not just the powers of -1. When carrying out calculations involving exponents and negative numbers the *sign* of the answer is determined from the *power*.

***Example** 1.6.5* Understand each of the following:

(a) $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = +(2^4) = 16$

Note that 4 is an even number so the answer is positive.

(b) $(-8)^3 = (-8) \times (-8) \times (-8) = -(8^3) = -512$

Note that 3 is an odd number so the answer is negative.

So far we have only looked at positive exponents. We will see negative exponents later in the course.

Questions

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1. Evaluate each of the following:

(**1**) $(-1)^{17}$

(**2**) $(-1)^{356}$

(**3**) $(-2)^3$

(**4**) 3^4

1.7 Square roots

The previous section on exponentiation explained how to square numbers. The **square root** of a number is a non-negative (that means positive or zero) number that when multiplied by itself (or *squared*), gives the original number. For example, the square root of 4 is 2, the square root of 25 is 5 and the square root of 0.81 is 0.9. The symbol used to denote square root is $\sqrt{\ }$, so $\sqrt{4} = 2$ and $\sqrt{0.81} = 0.9$. While there are two numbers which when squared give the value 4, namely 2 and -2, only one of these is called the square root of 4, namely $\sqrt{4} = 2$.

***Example** 1.7.1* $9 = 3^2$, so $3 = \sqrt{9}$. $16 = 4^2$, so $4 = \sqrt{16}$. $\sqrt{\frac{1}{4}} = \frac{1}{2}$, since $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

It is easy to show that a negative number **does not** have a square root. Recall back in multiplication of numbers on Page 9 we saw that a negative number can only be obtained by multiplying a positive number and a negative number (or vice versa) together. As we saw above, the square root of a number is a non-negative number that when multiplied by **itself** (or *squared*), gives the original number. Positive $\times$ negative is not multiplying by itself, so therefore negative numbers do not have square roots. (Check this on your calculator.)

- We can find the square root of numbers that are not integers. (We will see how to in the coming chapters.)
- A square root does not have to be an integer.
- An integer whose square root *does* happen to be an integer is called a *square number*.

***Example** 1.7.2*

- The first 11 square numbers are:

$$0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.$$

Questions

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1. Evaluate each of the following:

(1) $\sqrt{9}$

(2) $\sqrt{36}$

(3) $\sqrt{100}$

(4) $-\sqrt{64}$

(5) $\sqrt{0.04}$

(6) $\sqrt{0.0009}$ (You may wish to use a calculator for these last two.)

1.8 Order of operations

If we consider the expression $2 + 3 \times 5$, it is not clear how to arrive at the correct answer. By adding 2 to 3 and then multiplying by 5, we get 25; by multiplying 3 by 5 and then adding 2, we get 17. Which answer is correct? It is obvious that we need rules or a system to know in which **order** to perform different operations.

The word BEDMAS can help you to remember the rules. Each letter stands for a common mathematical operation; the **order** of the letters matches the **order** in which you should perform the mathematical operations.

letter	stands for:	example
B	brackets	$(3+4)$
E	exponentiation	3^4
D	division	$3 \div 4$
M	multiplication	3×4
A	addition	$3+4$
S	subtraction	$3-4$

- B: Look for any brackets in the expression, and evaluate inside the brackets **first**. If there are brackets inside brackets, then the **innermost** brackets get evaluated **before** the outermost ones.

- E: Next, any exponentiation must be evaluated.

- D, M: Next, evaluate divisions and multiplications. Note that even though D comes before M in BEDMAS, they have the same priority. This means that you evaluate any multiplication and division in the expression as they occur, working **from left to right**, before moving onto the next operations. *Note that the rule could also be called BEMDAS.*

- A, S: Finally, evaluate any additions or subtractions, working **from left to right**. Even though A comes before S in BEDMAS, they have the same priority, just like multiplication and division.

Example *1.8.1* Evaluate each of the following:

(a) $3 + 4 \times 2$ **(b)** $(3 + 4) \times 2$ **(c)** $12 \times 2 \div 3 \times 6 \div 12$

(d) $2 \times (1 + 4 \times (6 \div 3))$ **(e)** $3 + 2^{(2+1)}$ **(f)** $4 - 2^2 + (-3)^3$

Solutions:

(a)
$$\begin{aligned} 3+4\times 2 &= 3+8 && \times \text{ is evaluated first,}\\ &= 11 && \text{and then the } +. \end{aligned}$$

(b)
$$\begin{aligned} (3+4)\times 2 &= 7\times 2 && \text{Start with the brackets,}\\ &= 14 && \text{and then the } \times. \end{aligned}$$

(c)
$$\begin{aligned} 12\times 2\div 3\times 6\div 12 &= 24\div 3\times 6\div 12 && \text{Start with the } \times \text{ on the left,}\\ &= 8\times 6\div 12 && \text{then evaluate the } \div,\\ &= 48\div 12 && \text{next the } \times,\\ &= 4 && \text{then the remaining } \div. \end{aligned}$$

(d)
$$\begin{aligned} 2\times(1+4\times(6\div 3)) &= 2\times(1+4\times 2) && \text{Innermost brackets go first,}\\ &= 2\times(1+8) && \text{next the } \times \text{ in the remaining brackets,}\\ &= 2\times 9 && \text{then the } + \text{ in the brackets,}\\ &= 18 && \text{and finally the } \times. \end{aligned}$$

(e)
$$\begin{aligned} 3+2^{(2+1)} &= 3+2^3 && \text{Always evaluate brackets first,}\\ &= 3+8 && \text{next the exponent,}\\ &= 11 && \text{and then the } +. \end{aligned}$$

(f)
$$\begin{aligned} 4-2^2+(-3)^3 &= 4-4+(-3)^3 && \text{Start with the exponent on the left,}\\ &= 4-4+(-27) && \text{then the remaining exponent,}\\ &= 4-4-27 && \text{next simplify the brackets,}\\ &= 0-27 && \text{then perform the } - \text{ on the left,}\\ &= -27 && \text{and then the final } -. \end{aligned}$$

Note that in the last example there was nothing to evaluate within brackets. The brackets were asking us to take the exponent of -3 and not the exponent of 3. This differs to the exponent on the left which is asking you to subtract 2 to the power of 2. Be sure that you understand this subtle difference in the notation and how to interpret it.

Questions

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1. Answer each of the following questions, showing all working:

(**1**) Evaluate $12 \div 4 - 3$ and $12 \div (4 - 3)$

(**2**) Evaluate $90 \div 6 \div 3$ and $90 \div (6 \div 3)$

(**3**) Evaluate $5 \times 9 - 5$ and $5 \times (9 - 5)$

(**4**) Evaluate $3 \times 1 + 6$ and $3 \times (1 + 6)$

(**5**) Evaluate $48 \div 3 \times 4$ and $48 \div (3 \times 4)$

2. Answer each of the following questions, showing all working:

(**1**) Evaluate $8 \div 4 \times 2$

(**2**) Evaluate $2 + 4 \times 5 \div 2$

(**3**) Evaluate $3 \times (1 + 2) - 2 \times 4$

(**4**) Evaluate $2^3 - 4 \times 2 + (4 + 1)^2$

(**5**) Evaluate $10 - 6 + 3 \div 3 \times 2$

1.9 Factors and prime numbers

For any integer, we call another integer a **factor** of that integer if it divides exactly into that integer. For example, $14 = 2 \times 7$ and $14 = 1 \times 14$, so the factors of 14 are $1, 2, 7$ and 14. If two or more integers share the same factor, then we call that factor a **common factor** of those integers.

Finding the **factors** of relatively small integers is generally quite simple. However, when dealing with large integers (3 or more digits) the process becomes much more complicated. There are some simple tricks that let us easily check for small factors of an integer. An integer is divisible by:

2 if the last digit of the integer is even, for example, 2 is a factor of 298 as the last digit 8 is even;

3 if the sum of the digits in the integer is divisible by 3, for example, 3 is a factor of 195 as $1 + 9 + 5 = 15$ which is divisible by 3;

4 if the last two digits are an integer divisible by 4, for example, 4 is a factor of 2128 as the last two digits 28 form an integer divisible by 4;

5 if the integer ends in 0 or 5, for example, 5 is a factor of 875 as it ends in a 5;

6 if the integer is divisible by both 2 and 3, for example, 6 is a factor of 924 as the last digit 4 is even, meaning it is divisible by 2 and $9 + 2 + 4 = 15$, which is divisible by 3;

9 if the sum of the digits in the integer is divisible by 9, for example, 9 is a factor of 891 as $8 + 9 + 1 = 18$ which is divisible by 9;

10 if the integer ends in 0 and has more than 1 digit, for example, 10 is a factor of 510 as it ends in 0.

A **prime number** is a natural number whose only factors are 1 and itself. The first 7 prime numbers are $2,\ 3,\ 5,\ 7,$
$11,\ 13$ and 17. Since all even numbers are divisible by 2, the number 2 is the only even prime number. By convention, 1 is not considered to be a prime number. *Any* natural number larger than 1 is either prime, or can be written as a product of prime factors.

To write a number as a **product of prime factors** we can do the following:

- If the number is even, we divide the number by 2 (the first prime number) repeatedly until the resulting number is no longer divisible by 2. If you were able to divide by two, n times, then 2^n must be a factor of the original number. For example,

  $$\begin{aligned} 48 \div 2 &= 24 \\ 24 \div 2 &= 12 \\ 12 \div 2 &= 6 \\ 6 \div 2 &= 3 \end{aligned}$$

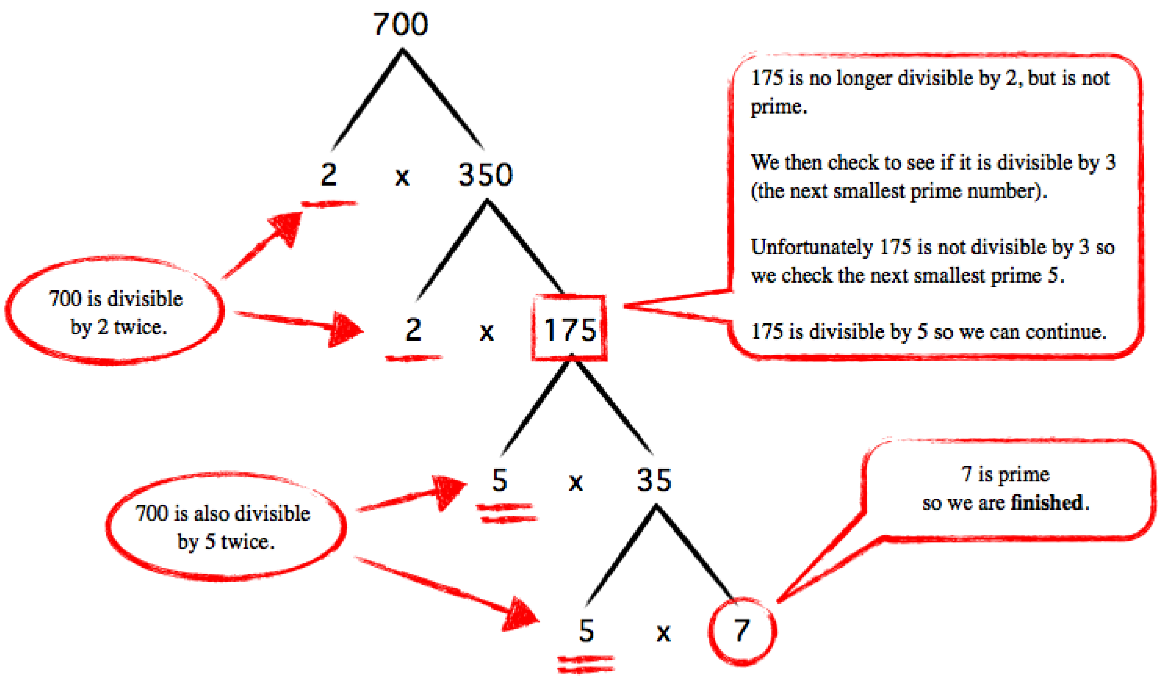
  3 is no longer divisible by 2 and we were able to divide 48 by 2 four times. This means that $2^4 = 2 \times 2 \times 2 \times 2$ is a factor of 48.

- Once we are unable to divide the number by 2 we check if the resulting number is prime. If it is, then we have completed the process. In the above example when we were no longer able to divide by 2 our resulting number was 3. 3 is a prime number so we have finished the process and can write 48 as a product of prime factors

  $$48 = 2 \times 2 \times 2 \times 2 \times 3 = 2^4 \times 3$$

  If the number is not prime, we repeat the above process with each successive prime number until the resulting number is a prime. This means we replace 2 with 3, and then 3 with 5, and then 5 with 7, etc. using the process above until the resultant number is prime.

***Example** 1.9.1* Write 700 as a product of prime factors.



So writing 700 as a product of primes, $2 \times 2 \times 5 \times 5 \times 7 \;=\; 2^2 \times 5^2 \times 7 \;=\; 700.$

Questions

1. Answer each of the following questions, showing all working:

(**1**) Is 14 a prime number? Why?

(**2**) Is 58 a prime number? Why?

(**3**) Is 15 a prime number? Why?

(**4**) Is 19 a prime number? Why?

(**5**) Is 8 a prime number? Why?

(**6**) Is 97 a prime number? Why?

2. Write each of the following as the product of prime factors (if it's not already prime):

(**1**) 12

(**2**) 31

(**3**) 48

1.10 Fractions: Multiplication and Division

We saw fractions at the beginning of this chapter when we introduced rational numbers. A fraction is the ratio of an integer and a non-zero integer, such as: $\frac{1}{2} = 1 \div 2$, $\frac{11}{3} = 11 \div 3$, and $-\frac{3}{4} = (-3) \div 4$ or equally $3 \div (-4)$. In a fraction we call the number on the top the **numerator**, and the number on the bottom the **denominator**.

Fractions are everywhere in life, for example, recipes, measurements and medicines, but working with them can be more challenging than working with integers. Evaluating $2 + 3$ is easy but how do we evaluate $\frac{1}{2} + \frac{3}{4}$ or $\frac{2}{3} \times \frac{1}{8}$? Let's start with multiplication of fractions as that is the easiest operation.

To multiply two fractions, we use the product of their numerators as the numerator of the result, and the product of their denominators as the denominator of the result. In other words, we simply multiply the two numbers on the top together to determine the numerator of the answer, and multiply the two numbers on the bottom to determine the denominator of the answer.

***Example** 1.10.1*

$$\frac{2}{3} \times \frac{1}{5} = \frac{2 \times 1}{3 \times 5} = \frac{2}{15}$$

In order to be able to **divide** fractions we must introduce inverses (or reciprocals). The inverse (or reciprocal) of a number is a number which you multiply the original number by to get an answer of 1. To obtain the inverse of a fraction you invert (or flip) the fraction. We can show this is true by multiplying two such fractions to obtain an answer of 1.

***Example** 1.10.2*

$$\frac{2}{3} \times \frac{3}{2} \;=\; \frac{2 \times 3}{3 \times 2} \;=\; \frac{6}{6} \;=\; 1$$

To divide two fractions, we **multiply** the first fraction in the expression by the **inverse** of the fraction after the division sign. In other words, we change the operation from division to multiplication and invert (or flip) the fraction we are dividing by.

***Example** 1.10.3*

$$\frac{3}{4} \div \frac{1}{8} \;=\; \frac{3}{4} \times \frac{\mathbf{8}}{\mathbf{1}} \;=\; \frac{3 \times 8}{4 \times 1} \;=\; \frac{24}{4} \;=\; 6$$

We call two fractions **equivalent fractions** if they are represented by the same point on the number line. Some examples of equivalent fractions are: $\frac{1}{2}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{19}{38}$. For any fraction, we can obtain an equivalent fraction by multiplying (or dividing) both the numerator and denominator by the **same** number. The two fractions remain equal because multiplying both the numerator and denominator by the same number is equivalent to multiplying the original fraction by 1 (and dividing both the numerator and denominator by the same number is equivalent to dividing the original fraction by 1). Note that any fraction of the form $\frac{n}{n}$ is equal to 1 (where both the n's are the same number and not zero).

***Example** 1.10.4*

$$\frac{1}{2} \;=\; \frac{1}{2} \times 1 \;=\; \frac{1}{2} \times \frac{3}{3} \;=\; \frac{1 \times 3}{2 \times 3} \;=\; \frac{3}{6}.$$

In this example, we have chosen to multiply the numerator and denominator by 3, but we could have chosen any non-zero number and still had a fraction equivalent to $\frac{1}{2}$. For example,

$$\frac{1}{2} \;=\; \frac{1}{2} \times 1 \;=\; \frac{1}{2} \times \frac{4}{4} \;=\; \frac{1 \times 4}{2 \times 4} \;=\; \frac{4}{8}.$$

When we obtain an answer that is a fraction, we need to check whether this fraction is in its simplest form. When a fraction is written in its **simplest form**, or **lowest terms**, the numerator and the denominator of the fraction have no common factors other than 1. This is done by identifying and then **cancelling** common factors of both the numerator and the denominator. To cancel a factor we divide the numerator and the denominator by the common factor; this eliminates it from the fraction *without changing the numerical value of the fraction*.

An effective method to find a fraction's simplest form is to first factorise both the numerator and denominator into primes (see Section 1.7) and then cancel primes factors common to both the numerator and the denominator.

Example *1.10.5* Consider the fraction $\frac{28}{42}$. Now 28 = $2 \times 2 \times 7$ and $42 = 2 \times 3 \times 7$, so the prime factors common to the numerator and denominator are 2 and 7. To write this fraction in its simplest form we need to cancel these prime factors.

$$\frac{28}{42} = \frac{28 \div 2}{42 \div 2} = \frac{14}{21} = \frac{14 \div 7}{21 \div 7} = \frac{2}{3}$$

An alternate and equivalent method, is to first identify and then cancel the highest common factor of both the numerator and denominator. This method is generally less efficient and leaves more margin for error then factorising into primes, but is often quicker when dealing with relatively small numbers.

Example *1.10.6* If we can quickly identify that 14 is the highest common factor of the numerator and denominator in the example above, then we can do the following:

$$\frac{28}{42} = \frac{28 \div 14}{42 \div 14} = \frac{2}{3}$$

The process of converting fractions to their simplest form is called **reducing to the lowest denominator**. To get full marks for your answers, you must always write fractions in their simplest form.

Sometimes you may see a question asking you to multiply an integer and a fraction, for example, $5 \times \frac{2}{3}$. This often confuses students but it's not hard. We saw in the first part of this chapter that an integer can be written as a rational number by placing it in a fraction over 1. So $5 = \frac{5}{1}$, making our question $\frac{5}{1} \times \frac{2}{3}$ which we can do.

You may have seen fractions written as mixed numbers (i.e. $3\,\frac{1}{2}$) in the past, but in this course (and in the majority of maths courses at university) we will avoid such representations and write $3\,\frac{1}{2}$ as $\frac{7}{2}$. It is easy to convert from one representation to another. For example, $3\,\frac{1}{2}$ is made up of 3 wholes and one half. In each whole there are two halves, so there are six halves in three wholes. Adding the initial one half gives us seven halves or $\frac{7}{2}$. To go the other way, we divide 7 by 2. 2 doesn't go evenly into 7, but we have 3×2 which is 6, so that's three wholes, then we need one more half to get to 7, so we have $3\,\frac{1}{2}$.

Questions

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1. Answer each of the following questions, showing all working:

(**1**) Find $\frac{0}{-16} \times \frac{-1}{-1}$

(**2**) Find $\frac{-8}{-7} \times \frac{14}{12}$

(**3**) Find $\frac{1}{17} \times \frac{3}{3}$

(**4**) Find $\frac{-13}{11} \times \frac{-13}{16}$

(**5**) Find $\frac{7}{8} \times \frac{-4}{3}$

(**6**) Find $\frac{-3}{4} \div \frac{4}{10}$

(**7**) Find $\frac{-10}{11} \div \frac{16}{-6}$

(**8**) Find $\frac{7}{17} \div \frac{7}{1}$

(**9**) Find $\frac{7}{4} \div \frac{8}{-12}$

(**10**) Find $\frac{11}{2} \div \frac{5}{-2}$

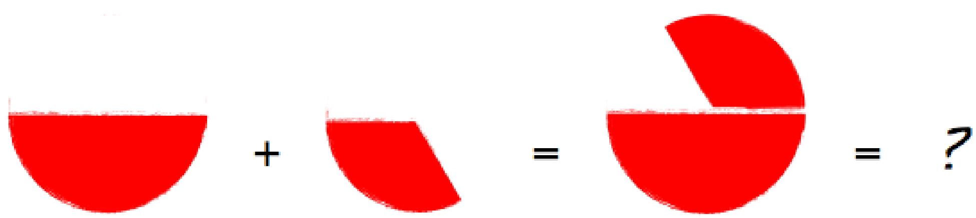
1.11 Fractions: Addition and subtraction

If two or more fractions have the same denominator then they are said to have a *common denominator*. **To add (or subtract)** fractions **with** common denominators, we add (or subtract) *just the numerators* and place the result above the common denominator.

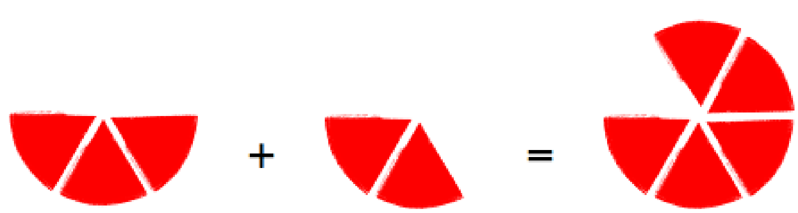
Example *1.11.1*

$$\frac{13}{32} + \frac{7}{32} - \frac{4}{32} = \frac{13 + 7 - 4}{32} = \frac{16}{32} = \frac{1}{2}$$

To add (or subtract) fractions **without** common denominators, we must convert each of the fractions to **equivalent fractions** so that the fractions we are adding (or subtracting) have common denominators. Once the fractions have common denominators we can proceed using the rule above. Consider a simple example of adding a half to a third, i.e. $\frac{1}{2} + \frac{1}{3}$.



It's clear the answer should be reasonably close to one, but what is the exact answer? If we rewrite the fractions into equivalent fractions with a common denominator; that is make it so that the fractions are in the same size pieces so we can easily compare (and count) them, the answer becomes much more obvious.



Example *1.11.2*

$$\begin{aligned}\frac{1}{2}+\frac{1}{3} &= \frac{1\times 3}{2\times 3}+\frac{1\times 2}{3\times 2}\\ &= \frac{3}{6}+\frac{2}{6}\\ &= \frac{3+2}{6}\\ &= \frac{5}{6}\end{aligned}$$

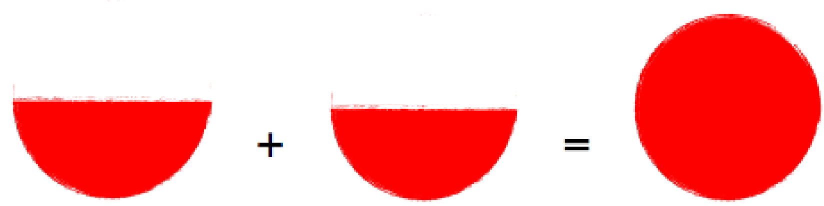
If we have two fractions to add (or subtract), the simplest common denominator to use is the product of the two denominators. For instance, to evaluate $\frac{2}{5}-\frac{7}{8}$, we could use a common denominator of $5\times 8=40$.

Example *1.11.3*

$$\begin{aligned}\frac{2}{5}-\frac{7}{8} &= \frac{2\times 8}{5\times 8}-\frac{7\times 5}{8\times 5}\\ &= \frac{16}{40}-\frac{35}{40}\\ &= -\frac{19}{40}\end{aligned}$$

A fraction is an interesting number in that it contains two numbers, one number on the top (the numerator), and one number on the bottom (the denominator). We need to be very careful when working with fractions. Although a fraction does contains two numbers it is in fact just **one number** and needs to be treated accordingly.

It is **very important** that you add and subtract fractions correctly. A very common error people make when adding (or subtracting) fractions is to add (or subtract) the denominators. Let's consider an easy example and add two halves. As we saw above $\frac{1}{2}+\frac{1}{2}=\frac{1+1}{2}=\frac{2}{2}=1$, and this appears logical if we represent the addition visually.



If we were to make an error and perform the addition incorrectly and add the denominators together as well as adding the numerators, i.e. $\frac{1}{2}+\frac{1}{2}=\frac{1+1}{2+2}=\frac{2}{4}=\frac{1}{2}$, the visual representation as well as the answer make absolutely no sense.

From the diagram it's quite apparent that adding fractions in this way is incorrect and that we need to be careful **not** to add denominators together.

Although the majority of calculations involving fractions are generally straightforward, they can become troublesome when numerators and denominators include letters and mathematical expressions (as you will see later in the course). Fractions are used quite a lot in this course, so take the time now to practise on some examples.

Questions

1. Answer each of the following questions, showing all working:

(**1**) Find $\dfrac{7}{2} + \dfrac{-8}{5}$

(**2**) Find $\dfrac{13}{18} + \dfrac{6}{17}$

(**3**) Find $\dfrac{0}{14} + \dfrac{-4}{-6}$

(**4**) Find $\dfrac{14}{8} + \dfrac{15}{8}$

(**5**) Find $\dfrac{-9}{14} + \dfrac{-3}{13}$

(**6**) Find $\dfrac{1}{3} - \dfrac{-3}{7}$

(**7**) Find $\dfrac{8}{3} - \dfrac{2}{15}$

(**8**) Find $\dfrac{-9}{-4} - \dfrac{-1}{9}$

(**9**) Find $\dfrac{-14}{10} - \dfrac{12}{8}$

(**10**) Find $\dfrac{13}{12} - \dfrac{-8}{20}$

2. Answer each of the following questions, showing all working. Remember to apply BEDMAS.

(**1**) Find $\dfrac{0}{4} \div \dfrac{-35}{18} + \dfrac{53}{-38} - \dfrac{-7}{-38}$

(**2**) Find $\dfrac{4}{-1} - \dfrac{1}{-15} + \dfrac{-1}{15} + \dfrac{-8}{13}$

(**3**) Find $\dfrac{-8}{6} \div \dfrac{56}{-29} \div \dfrac{58}{27} \div \dfrac{36}{38}$

(**4**) Find $\dfrac{6}{-8} \times \dfrac{-45}{-12} + \dfrac{-60}{-16} - \dfrac{-36}{-48}$

(**5**) Find $\left(\dfrac{-8}{8} - \dfrac{-60}{56}\right) \times \dfrac{-3}{6} - \dfrac{-18}{-14}$

3. There are some very common errors when dealing with fractions. Each of the following examples is **incorrect**. In each case, work out the correct answer. (Parts (b) and (c) are particularly common errors.)

(1) $\frac{1}{2} + \frac{3}{4} = \frac{1+3}{4+2} = \frac{4}{6}$

(2) $\frac{4+2}{2} = 4$

(3) $\frac{6+4}{1+4} = \frac{6}{1} = 6$

(4) $\frac{2}{5} \times 3 = \frac{2 \times 3}{5 \times 3} = \frac{6}{15}$

(5) $\frac{2}{2} = 0$

(Remember, these are all incorrect!)

Fully worked solutions to all the questions can be found on the MATH1040 website

- www.maths.uq.edu.au/MATH1040.