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THE UNIVERSITY OF QUEENSLAND

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First Semester Examination, 2006

# MATH1050

## Mathematical Foundations

**Time:** **Two hours** for working

Ten minutes for perusal before examination begins

**CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION SCRIPT.**

Use the back pages if the space provided is insufficient.
Candidates should attempt ALL questions. All questions carry the indicated number of marks. There are 60 marks in total. Check that this examination paper has 23 printed pages.

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**MATH1050 — Mathematical Foundations**
**First Semester Examination, 2006 (continued)**

| **EXAMINER'S USE ONLY** | | | |
|---|---|---|---|
| QUESTION | MARK | QUESTION | MARK |
| 1-10 | | 17 | |
| 11 | | 18 | |
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REFERENCE PAGE

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$.

If $\det(A) \neq 0$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, then $||\,\mathbf{v}\,|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ and $\hat{\mathbf{v}} = \dfrac{1}{||\,\mathbf{v}\,||}\mathbf{v}$.

Scalar product: $\mathbf{v}.\mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$ .

Vector product: $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Cosine Law: $a^2 = b^2 + c^2 - 2bc\cos A$ Sine Law: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Series: $\displaystyle\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$, $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

The definition of the derivative $f'(a)$ of a function $f$ at the number $a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Differentiation rules:

$$(fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad (f \circ g)'(x) = f'(g(x)).g'(x).$$

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Trigonometric identities:

$$\begin{aligned} \sin(-\theta) &= -\sin(\theta) \\ \cos(-\theta) &= \cos(\theta) \\ \sin(\theta+\phi) &= \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) \\ \cos(\theta+\phi) &= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \end{aligned}$$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

De Moivre's theorem: If $z = r(\cos(\theta) + i\sin(\theta))$ then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

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For each of the following 10 multiple choice questions, write the letter corresponding to the correct answer in the space provided at the end of the question. Each correct answer is worth 0.5 marks, each incorrect answer is worth 0 marks.

1. Which of the following statements is true ? *b*

   (a) $\sin\left(\theta + \frac{\pi}{2}\right) = \sin(\theta)$

   (b) $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$

   (c) $\sin\left(\theta + \frac{\pi}{2}\right) = \tan(\theta)$

   (d) none of the above

2. If $|x| < 2$ which of the following statements is true? *d*

   (a) $0 < x \leq 2$

   (b) $-2 \leq x \leq 2$

   (c) $-2 < x \leq 2$

   (d) $-2 < x < 2$

3. Which of the following intervals corresponds to the inequality $|-x| > 3$? *a*

   (a) $(-\infty, -3) \cup (3, \infty)$

   (b) $(-\infty, -3] \cup [3, \infty)$

   (c) $(-3, 3)$

   (d) $[-3, 3]$

4. The domain of the function $f(x) = a - x^2$ is: *a*

   (a) $(-\infty, \infty)$

   (b) $(-a, \infty)$

   (c) $(0, \infty)$

   (d) $[-a, \infty)$

5. The range of the function $f(x) = a - x^2$ is: *d*

   (a) $(-\infty, \infty)$

   (b) $(-\infty, a)$

   (c) $(0, \infty)$

   (d) $(-\infty, a]$

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6. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be any vectors in $\mathbb{R}^3$. Which of the following statements is generally false? *c*

   (a) $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$
   (b) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
   (c) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$
   (d) none of the above

7. If $S = 2x - 4x^2 + 6x^3 - 8x^4$, which of the following statements is true? *a*

   (a) $S = 2\sum_{k=0}^{3}(-1)^k(k+1)x^{k+1}$
   (b) $S = 2\sum_{k=1}^{4}(-1)^{k+1}(k+1)x^k$
   (c) $S = 2\sum_{k=0}^{3}(-1)^{(k+1)}(k+1)x^{k+1}$
   (d) none of the above

8. If $A$ and $B$ are invertible $n \times n$ matrices and $C = AB$, which of the following statements is true? *b*

   (a) $C$ is singular
   (b) $C$ non-singular
   (c) $C$ is diagonal
   (d) none of the above.

9. If $y = x^3 + 2x^2 + 4x$ then $\dfrac{dy}{dx}$ is: *b*

   (a) $x^2 + 2x + 4$
   (b) $3x^2 + 4x + 4$
   (c) $3x + 4$
   (d) $3x^2 + 4x$

10. If $y = x^3 + 2x^2 + 4x$ then $\int y\,dx$ is: *d*

    (a) $\dfrac{1}{4}x^4 + \dfrac{4}{3}x^3 + x^2 + c$
    (b) $x^4 + \dfrac{4}{3}x^3 + 2x^2 + x$
    (c) $x^4 + \dfrac{2}{3}x^3 + x^2 + x$
    (d) $\dfrac{1}{4}x^4 + \dfrac{2}{3}x^3 + 2x^2 + c$

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**MATH1050 — Mathematical Foundations**
**First Semester Examination, 2006 (continued)**

11. Ticket prices for the Livid Festival are \$90 full price and \$70 for concessions. The total number of people attending is 35,000, and the takings from ticket sales are \$2,884,960. Set up a pair of simultaneous equations and solve them using matrices in order to determine how many of each type of ticket were sold. (3 marks)

Solution
Let $x$ denote the number of full price tickets sold and $y$ the number of concession tickets sold. Then

$$\begin{aligned} x + y &= 35,000 \\ 90x + 70y &= 2,884,960 \end{aligned}$$

As a matrix equation we have

$$\begin{pmatrix} 1 & 1 \\ 90 & 70 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 35,000 \\ 2,884,960 \end{pmatrix}.$$

(1 mark)

$$\begin{aligned} \begin{vmatrix} 1 & 1 \\ 90 & 70 \end{vmatrix} &= 70 - 90 = -20 \\ \begin{pmatrix} 1 & 1 \\ 90 & 70 \end{pmatrix}^{-1} &= \frac{-1}{20} \begin{pmatrix} 70 & -1 \\ -90 & 1 \end{pmatrix}. \end{aligned}$$

(1 mark)
Hence

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{-1}{20} \begin{pmatrix} 70 & -1 \\ -90 & 1 \end{pmatrix} \begin{pmatrix} 35,000 \\ 2,884,960 \end{pmatrix} \\ &= \frac{-1}{20} \begin{pmatrix} 70 \times 35,000 - 2,884,960 \\ -90 \times 35,000 + 2,884,960 \end{pmatrix} \\ &= \begin{pmatrix} 21,748 \\ 13,252 \end{pmatrix}. \end{aligned}$$

There were 21,748 full price tickets sold and 13,252 concession tickets sold. (1 mark)

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12. A triangle has vertices in three-dimensional space given by the points $A = (0, 0, 0)$, $B = (2, 0, 3)$ and $C(0, 1, 0)$. Below, give all answers correct to two decimal places.

    (a) Calculate $|| \vec{AB} ||$, $|| \vec{BC} ||$ and $|| \vec{AC} ||$. (2 marks)

    (b) Calculate the angle at $A$. (1 mark)

    (c) Using the vector cross product, find the area of the triangle. (2 marks)

    Solution

    (a)

$$\begin{aligned}
\vec{AB} &= \mathbf{b} - \mathbf{a} = 2\mathbf{i} + 3\mathbf{k} \\
|| \vec{AB} || &= \sqrt{2^2 + 3^2} = \sqrt{13} \approx 3.61. \qquad (0.5 \text{ marks}) \\
\vec{BC} &= \mathbf{c} - \mathbf{b} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k} \qquad (0.5 \text{ marks}) \\
|| \vec{BC} || &= \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14} \approx 3.74. \qquad (0.5 \text{ marks}) \\
\vec{AC} &= \mathbf{c} - \mathbf{a} = \mathbf{j} \\
|| \vec{AC} || &= 1. \qquad (0.5 \text{ marks})
\end{aligned}$$

    (b) Using the cosine law,

$$\begin{aligned}
\cos A &= \frac{|| \vec{AB} ||^2 + || \vec{AC} ||^2 - || \vec{BC} ||^2}{2 || \vec{AB} || \, . \, || \vec{AC} ||} \\
&= \frac{13 + 1 - 14}{2\sqrt{13}} \\
&= 0, \\
A &= 90^0 \text{ or } \frac{\pi}{2} \qquad (1 \text{ mark})
\end{aligned}$$

    (Alternatively, $\vec{AB}.\vec{AC} = 0 \implies A = \frac{\pi}{2}$.)

    (c) Note: The question specifically asks to use the cross product. Half marks should be given if an alternative method is used.

$$\begin{aligned}
\vec{AB} \times \vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 0 & 1 & 0 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\
&= -3\mathbf{i} + 2\mathbf{k}. \\
\text{area} &= \frac{1}{2} || \vec{AB} \times \vec{AC} || \\
&= \frac{1}{2} \sqrt{3^2 + 2^2} \\
&= \frac{1}{2} \sqrt{13} \approx 1.80
\end{aligned}$$

13. Uschi and Hans go roller skating. Hans has a mass of 70 kg, and skates at a constant speed of 10 m/s. Uschi has a mass of 50 kg, is carrying a bag with mass 5 kg, and skates at a constant speed of 6 m/s. They collide at an angle of 140 degrees. During the collision, Hans accidently grabs Uschi's bag. If Hans and the bag are at rest after the collision, use the conservation of momentum to determine the speed of Uschi immediately after the collision. (4 marks)

Solution
There should be a diagram. (0.5 marks)
Let Uschi travel in the direction of the $x$-axis, so the initial velocity of Uschi is

$$\mathbf{v}_U^i = 6\mathbf{i}$$

and the initial momentum of Uschi (and the bag) is

$$\mathbf{p}_U^i = 55 \times 6\mathbf{i} = 330\mathbf{i}. \qquad \text{(0.5 marks)}$$

The initial velocity of Hans is

$$\begin{aligned} \mathbf{v}_H^i &= 10(\cos(140)\mathbf{i} + \sin(140)\mathbf{j}) \\ &\approx -7.66\mathbf{i} + 6.43\mathbf{j} \end{aligned}$$

so the initial momentum of Hans is

$$\begin{aligned} \mathbf{p}_H^i &\approx 70(-7.66\mathbf{i} + 6.43\mathbf{j}) \\ &\approx -536\mathbf{i} + 450\mathbf{j}. \qquad \text{(0.5 marks)} \end{aligned}$$

The total initial momentum is

$$\begin{aligned} \mathbf{p}^i &= \mathbf{p}_U^i + \mathbf{p}_H^i \\ &\approx -206\mathbf{i} + 450\mathbf{j}. \qquad \text{(0.5 marks)} \end{aligned}$$

By the conservation of momentum, the total final momentum is equal to the total initial momentum. Since Hans is at rest after the collision, $\mathbf{p}_H^f = \mathbf{0}$. Therefore

$$\begin{aligned} \mathbf{p}_U^f &= \mathbf{p}_U^i + \mathbf{p}_H^i \approx -206\mathbf{i} + 450\mathbf{j}. \\ ||\,\mathbf{p}_U^f\,|| &\approx \sqrt{206^2 + 450^2} \approx 495 \qquad \text{(0.5 marks)} \\ ||\,\mathbf{v}_U^f\,|| &= \frac{1}{50}\,||\,\mathbf{p}_U^f\,|| \approx 9.90 \qquad \text{(0.5 marks)} \end{aligned}$$

After the collision, the speed of Uschi is 9.90 m/s.

14. (a) Recall that a closed formula for a geometric sequence is of the form $a_k = ar^{k-1}$ where $a$ is the first term and $r$ is the common ratio. Defining the series

$$S_n = \sum_{k=1}^{n} a_k,$$

prove **by induction** that

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

(3 marks)

(b) An exponential population model has the form $P_n = P_0(1 + k)^n$ where $k$ is the growth rate, $P_0$ is the initial population and $P_n$ is the population after $n$ years. If a particular species of cockroaches has an initial population of 10, and a population of 10,000 five years later, determine the growth rate $k$ (give answer to two decimal places). (1 mark)

Solution

(a) Let $P(n) = \frac{a(r^n - 1)}{r - 1}$. Then $P(1) = \frac{a(r - 1)}{r - 1} = a$, and $S_1 = a_1 = a$. Therefore $P(1) = S_1$. (1 mark)
Now

$$\begin{aligned}
S_n &= S_{n-1} + a_n \\
&= S_{n-1} + ar^{n-1} \\
&= P(n-1) + ar^{n-1} \qquad \text{(inductive assumption) — (1 mark)} \\
&= \frac{a(r^{n-1} - 1)}{r - 1} + ar^{n-1} \\
&= \frac{ar^{n-1} - a + ar^{n-1}(r - 1)}{r - 1} \\
&= \frac{ar^{n-1} - a + ar^n - ar^{n-1}}{r - 1} \\
&= \frac{a(r^n - 1)}{r - 1} \\
&= P(n) \qquad \text{(1 mark)}
\end{aligned}$$

By induction, $S_n = \frac{a(r^n - 1)}{r - 1}$.

(b) Rearranging the formula gives

$$\begin{aligned}(1+k)^n &= \frac{P_n}{P_0} \\ 1+k &= \left(\frac{P_n}{P_0}\right)^{\frac{1}{n}} \\ k &= \left(\frac{P_n}{P_0}\right)^{\frac{1}{n}} - 1.\end{aligned}$$

Now $n = 5$, $P_0 = 10$, $P_5 = 10,000$. Substituting in these values

$$k = \left(\frac{10,000}{10}\right)^{\frac{1}{5}} - 1 \approx 2.98.$$

The growth rate is approximately 2.98. (1 mark).

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15. The first six terms of an infinite *polynomial* sequence $\{a_n\}_{n=1}^{\infty}$ are given by

$$a_1 = -1,\ a_2 = -2,\ a_3 = -1,\ a_4 = 2,\ a_5 = 7,\ a_6 = 14.$$

(a) Use the method of finite differences to determine a closed form for the sequence. (4 marks)

(b) From the result of part (a), determine a recursive formula for the sequence. (1 mark)

Solution

(a) Calculating the first differences:

$$b_1 = a_2 - a_1 = -1,\ b_2 = a_3 - a_2 = 1,\ b_3 = a_4 - a_3 = 3,\ b_4 = a_5 - a_4 = 5,\ b_5 = a_6 - a_5 = 7.$$

(1 mark)
Calculating the second differences:

$$c_1 = b_2 - b_1 = 2,\ c_2 = b_3 - b_2 = 2,\ c_3 = b_4 - b_3 = 2,\ c_4 = b_5 - b_4 = 2.$$

(1 mark)
Since the second differences are the same, we look for a second order polynomial solution

$$a_n = An^2 + Bn + C.$$

(1 mark)
Now

$$\begin{aligned} -1 &= a_1 = A + B + C, \\ -2 &= a_2 = 4A + 2B + C, \\ -1 &= a_3 = 9A + 3B + C. \end{aligned}$$

Solving the above equations simultaneously we find the solution

$$A = 1,\ B = -4,\ C = 2.$$

The closed form formula is $a_n = n^2 - 4n + 2$.
(1 mark)

(b)

$$a_{n-1} = (n-1)^2 - 4(n-1) + 2 = n^2 - 6n + 7,$$

$$a_n - a_{n-1} = 2n - 5,$$

Therefore the recursive formula is $a_n = a_{n-1} + 2n - 5$ with $a_1 = -1$.
(1 mark)

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16. For this question, give all answers with rational denominators.

(a) Find the exact value of $\cos\left(\frac{5\pi}{12}\right)$ (where the angle is measured in radians). (2 marks)

(b) Find the exact value of $\cos\left(\frac{5\pi}{24}\right)$. (2 marks)

Solution

(a)

$$\begin{aligned}
\cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{2\pi}{12}+\frac{3\pi}{12}\right) \\
&= \cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right) && \text{(0.5 mark)} \\
&= \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) && \text{(0.5 mark)} \\
&= \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}-\frac{1}{2}.\frac{1}{\sqrt{2}} && \text{(0.5 mark)} \\
&= \frac{1}{2\sqrt{2}}\left(\sqrt{3}-1\right) \\
&= \frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right). && \text{(0.5 mark)}
\end{aligned}$$

(b) From the formula sheet

$$\begin{aligned}
\cos(\theta+\phi) &= \cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi), \\
\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta) \\
&= 2\cos^2(\theta)-1, \\
\cos(\theta) &= \sqrt{\frac{1}{2}(1+\cos(2\theta))} \qquad \text{(1 mark).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\cos\left(\frac{5\pi}{24}\right) &= \sqrt{\frac{1}{2}\left(1+\cos\left(\frac{5\pi}{12}\right)\right)} \\
&= \sqrt{\frac{1}{2}\left(1+\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)\right)} \\
&= \sqrt{\frac{2}{16}\left(4+\sqrt{2}\left(\sqrt{3}-1\right)\right)} \\
&= \frac{1}{4}\sqrt{2\left(4+\sqrt{6}-\sqrt{2}\right)}. \qquad \text{(1 mark)}
\end{aligned}$$

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17. Solve the following inequality. Write your solution in interval notation and illustrate it on a number line.

$$\frac{|y-4|}{y+2} < 3.$$

(4 marks)

Solution

First we note $y \neq 2$. Now for $y \geq 4$ we have $y + 2 > 0$, then

$$\begin{aligned} y - 4 &< 3(y+2) \\ -10 &< 2y \\ -5 &< y \end{aligned}$$

so $y \in [4, \infty)$. (1 mark)

Next consider $-2 < y < 4$. Then $y - 4 < 0$ and $y + 2 > 0$ so that

$$\begin{aligned} -y + 4 &< 3(y+2) \\ -2 &< 4y \\ -\frac{1}{2} &< y \end{aligned}$$

giving $y \in \left(-\frac{1}{2}, 4\right)$. (1 mark)

For $y < -2$ we get $y - 4 < 0$ and $y + 2 < 0$ such that

$$\begin{aligned} -y + 4 &> 3(y+2) \\ -2 &> 4y \\ -\frac{1}{2} &> y \end{aligned}$$

so $y \in (-\infty, -2)$. (1 mark)

The complete solution is $y \in (-\infty, -2) \cup \left(-\frac{1}{2}, \infty\right)$.

(1 mark for the diagram)

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**MATH1050 — Mathematical Foundations**
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18. Let $f(x) = \sqrt{2x+1}$ and let $g(x) = \dfrac{1}{x^2}$.

(a) State the domain and range of $f$. (1 mark)

(b) State the domain and range of $g$. (1 mark)

(c) Determine $(g \circ f)(x)$ and state its domain and range. (2 marks)

(d) Is $g$ a one-to-one function? Explain your answer. (1 mark)

Solution

(a) The domain of $f$ is $\left[-\frac{1}{2}, \infty\right)$ and the range is $[0, \infty)$. (1 mark)

(b) The domain of $g$ is $\mathbb{R}\backslash\{0\}$ and the range is $(0, \infty)$. (1 mark)

(c) $(g \circ f)(x) = g(f(x)) = g(\sqrt{2x+1}) = \dfrac{1}{2x+1}$ (1 mark)
Domain of $(g \circ f)(x)$ are those values of $x$ in the domain of $f$ such that $\sqrt{2x+1}$ is in the domain of $g$. Require $x > -\frac{1}{2}$ (non-zero, real denominator). Then domain of $(g \circ f)(x)$ is $\left(-\frac{1}{2}, \infty\right)$. Range of $(g \circ f)(x)$ is $(0, \infty)$. (1 mark)

(d) $g(x)$ is not $1-1$ as $g(x) = g(-x)$. (1 mark)

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19. Let $f(x) = \dfrac{3}{x-2}$.

(a) Use the definition of the derivative as a limit to determine the slope of the tangent line at $x = -1$. (3 marks)

(b) Find the equation of the tangent line to $f$ at $x = -1$. (2 marks)

Solution

(a)

$$\begin{aligned} f'(-1) &= \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{-1+h-2} - \frac{3}{-3}}{h} \qquad \text{(1 mark)} \\ &= \lim_{h\to 0} \frac{\frac{3}{h-3} + 1}{h} \\ &= \lim_{h\to 0} \frac{1}{h}.\frac{3+h-3}{h-3} \qquad \text{(1 mark)} \\ &= \lim_{h\to 0} \frac{1}{h-3} \\ &= -\frac{1}{3} \qquad \text{(1 mark)} \end{aligned}$$

which is

(b) The slope of the tangent line at $x = -1$ is $\dfrac{-1}{3}$, and $f(-1) = -1$. (1 mark)
Hence the equation of the tangent line to $f$ at $x = -1$ is given by

$$\begin{aligned} y + 1 &= -\frac{1}{3}(x+1) \\ y &= -\frac{1}{3}x - \frac{4}{3}. \qquad \text{(1 mark)} \end{aligned}$$

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20. (a) Let $z = \sqrt{3} - i$. Use De Moivre's Theorem to evaluate $z^3$ and express your answer in polar form. (2 marks)

    (b) One solution of $2z^3 - 9z^2 + 30z - 13 = 0$ is $z = 2 - 3i$. Find all the other solutions of this equation over $\mathbb{C}$. (2 marks)

    (c) Determine all values of $z \in \mathbb{C}$ such that $z^3 = -27i$. Illustrate your values of $z$ on an Argand diagram. (3 marks)

Solution

(a) $|z| = 2, \quad \text{Arg}(z) = -\dfrac{\pi}{6}$.

In polar form $z = 2 \text{ cis}\left(-\dfrac{\pi}{6}\right)$. (1 mark)

Now $z^3 = 2^3 \text{ cis}\left(-\dfrac{3\pi}{6}\right) = 8 \text{ cis}\left(-\dfrac{\pi}{2}\right)$. (1 mark)

(b) If $z = 2 - 3i$ is a solution then so is $z = 2 + 3i$. Next write

$$\begin{aligned} (az+b)(z-(2-3i))(z-(2+3i)) &= 0. \qquad \text{(1 mark)} \\ (az+b)(z^2-4z+13) &= 0 \\ az^3 + (b-4a)z^2 + (13a-4b)z + 13b &= 0 \end{aligned}$$

Comparing with the original polynomial, we see $a = 2, b = -1$. The third solution is $z = \dfrac{1}{2}$. (1 mark)

(c)

$$\begin{aligned} z^3 &= -27i \\ &= 27 \text{ cis}\left(-\frac{\pi}{2}\right) \\ &= 27 \text{ cis}\left(-\frac{\pi}{2} + 2k\pi\right), \quad k \in \mathbb{Z} \qquad \text{(1 mark)} \end{aligned}$$

then

$$\begin{aligned} z &= 3 \text{ cis}\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right), \quad k \in \mathbb{Z} \qquad \text{(1 mark)} \\ &= 3 \text{ cis}\left(-\frac{\pi}{6}\right),\ 3 \text{ cis}\left(\frac{\pi}{2}\right),\ 3 \text{ cis}\left(\frac{7\pi}{6}\right) \qquad \text{(1 mark)} \end{aligned}$$

This last one can be written in terms of the principal argument as $3 \text{ cis}\left(-\dfrac{5\pi}{6}\right)$.

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21. Determine the derivative for the following function.

(a) $g(u) = 4u^2\ e^{2u}$. (2 marks)

Evaluate the following indefinite integral.

(b) $\int x^2\sqrt{2x^3+4}\ dx$ (2 marks)

Solution

(a)

$$\begin{aligned} g'(u) &= 8ue^{2u} + 8u^2e^{2u} \\ &= 8u(u+1)e^{2u}. \end{aligned}$$

(1 mark for use of product rule, 1 mark for use of chain rule)

(b) Use substitution. Put $u = 2x^3 + 4$. Then $du = 6x^2dx$. (1 mark)
The integral becomes

$$\begin{aligned} \int x^2\sqrt{2x^3+4}\ dx &= \int \frac{1}{6}u^{\frac{1}{2}}\ du \\ &= \frac{1}{6}.\frac{2}{3}.u^{\frac{3}{2}} + C \\ &= \frac{1}{9}(2x^3+4)^{\frac{3}{2}} + C. \end{aligned}$$

(1 mark)

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22. Let $f(x) = -3x^2 + 2x$.

(a) Evaluate the Riemann sum for $f(x)$, with $0 \leq x \leq 2$, having 20 subintervals and taking the sample points to be the right endpoints. (3 marks)

(b) Evaluate the definite integral $\int_0^2 f(x)\ dx$. (2 marks)

Solution

(a)

$$A_R = \sum_{i=1}^{n} f(x_i)\Delta x. \qquad \text{(0.5 marks)}$$

$$\Delta x = \frac{2-0}{20} = \frac{1}{10}. \qquad \text{(0.5 marks)}$$

$$x_i = 0 + \frac{i}{10}. \qquad \text{(0.5 marks)}$$

Then

$$\begin{aligned} f(x_i) &= -3x_i^2 + 2x_i \\ &= -3\frac{i^2}{100} + 2\frac{i}{10} \qquad \text{(0.5 marks)} \end{aligned}$$

Using the summation formulae given on the reference page,

$$\begin{aligned} A_R &= \sum_{i=1}^{20} \left(-\frac{3i^2}{100} + \frac{2i}{10}\right)\frac{1}{10} \\ &= -\frac{3}{1000}\sum_{i=1}^{20} i^2 + \frac{2}{100}\sum_{i=1}^{20} i \qquad \text{(0.5 marks)} \\ &= -\frac{3}{1000}.\frac{20\times 21\times 41}{6} + \frac{2}{100}.\frac{20\times 21}{2} \\ &= -\frac{21\times 41}{100} + \frac{21}{5} \\ &= -4.4. \qquad \text{(0.5 marks)} \end{aligned}$$

(b)

$$\begin{aligned} \int f(x)\ dx &= \int_0^2 (-3x^2 + 2x)\ dx \\ &= \left[-x^3 + x^2\right]_0^2 \qquad \text{(1 mark)} \\ &= -8 + 4 \\ &= -4. \qquad \text{(1 mark)} \end{aligned}$$

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