11. Given matrices $A = \begin{pmatrix} 1 & 4 & 0 \\ -2 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 & -2 \\ 1 & 3 & 0 \end{pmatrix}$, evaluate the following where possible:

(a) $A + B$ (1 marks)

(b) $A + B^T$ (1 mark)

(c) AB (1 mark)

(d) $A^T B$ (2 marks)

(a) $A+B = \begin{pmatrix} 1 & 4 & 0 \\ -2 & 3 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 3 & -2 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 7 & -2 \\ -1 & 6 & 1 \end{pmatrix}$

(b) As A is 2×3 and B^T is 3×2, $A+B^T$ is not defined

(c) As A is 2×3 and B is 2×3, AB is not defined

(d) $A^T = \begin{pmatrix} 1 & -2 \\ 4 & 3 \\ 0 & 1 \end{pmatrix}$ (1 mark)

$A^T B = \begin{pmatrix} 1 & -2 \\ 4 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -2 \\ 1 & 3 & 0 \end{pmatrix}$

$= \begin{pmatrix} -1-2 & 3-6 & -2+0 \\ -4+3 & 12+9 & -8+0 \\ 0+1 & 0+3 & 0+0 \end{pmatrix} = \begin{pmatrix} -3 & -3 & -2 \\ -1 & 21 & -8 \\ 1 & 3 & 0 \end{pmatrix}$

(1 mark)

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12. Find the values of x and y such that $\det(A) = 0$ where

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & 3 & x \\ y & 0 & 2 \end{pmatrix}.$$

(2 marks)

Expanding about the first row

$$\det(A) = 1 \times \begin{vmatrix} 3 & x \\ 0 & 2 \end{vmatrix} - 0 \times \begin{vmatrix} -2 & x \\ y & 2 \end{vmatrix} - 2 \times \begin{vmatrix} -2 & 3 \\ y & 0 \end{vmatrix}$$ (1 mark)

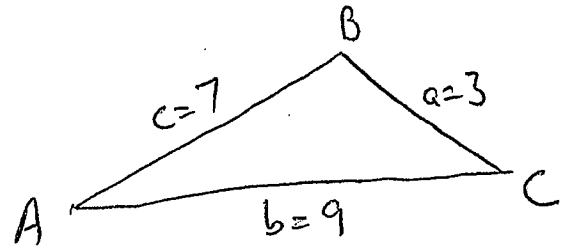
$$= 6 + 0 - 2 \times (-3y)$$

$$= 6 + 6y$$

Hence $\det(A) = 0$ when $y = -1$ and x can take any value (1 mark)

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13. A triangle has sides of length 3,7,9. Determine the three angles of the triangle, measured in degrees. (Give answers to an accuracy of two decimal places) (3 marks)



Using the Cosine Law,

$a^2 = b^2 + c^2 - 2bc\cos A.$

$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{81+49-9}{2\times 9\times 7} = \frac{121}{126}$

$A \approx 16.20^\circ$ (1 mark)

$b^2 = a^2 + c^2 - 2ac\cos B$

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{9+49-81}{2\times 3\times 7} = \frac{-23}{42}$

$B \approx 123.20^\circ$ (1 mark)

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{9+81-49}{2\times 3\times 9} = \frac{41}{54}$

$C \approx 40.60^\circ$ (1 mark)

(Could have also used the Sine Law)

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MATH1050 — Mathematical Foundations
Mid Semester Examination, April 7-10, 2006 (continued)

14. Let **u** be the vector which has magnitude 8 and direction $\frac{5\pi}{4}$. Let $\mathbf{v} = 3\mathbf{i} - 3\sqrt{3}\mathbf{j}$. Evaluate **u**.**v**. (Give answer to an accuracy of two decimal places.) (2 marks)

$$\underset{\sim}{u} = 8\cos\left(\tfrac{5\pi}{4}\right)\underset{\sim}{i} + 8\sin\left(\tfrac{5\pi}{4}\right)\underset{\sim}{j}$$
$$= -8\cos\left(\tfrac{\pi}{4}\right)\underset{\sim}{i} - 8\sin\left(\tfrac{\pi}{4}\right)\underset{\sim}{j}$$
$$= -4\sqrt{2}\,\underset{\sim}{i} - 4\sqrt{2}\,\underset{\sim}{j} \qquad (1 \text{ mark})$$

$$\underset{\sim}{u}.\underset{\sim}{v} = \left(-4\sqrt{2}\right)\times 3 + \left(-4\sqrt{2}\right)\times\left(-3\sqrt{3}\right)$$
$$= 12\sqrt{2}\left(\sqrt{3}-1\right)$$
$$\approx 12.42 \qquad (1 \text{ mark})$$

Alternatively, $\|\underset{\sim}{v}\| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{36} = 6$

$$\phi = \arctan\left(\frac{3\sqrt{3}}{3}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$$

Since $\underset{\sim}{v}$ is in the 4th quadrant, $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

Thus $\underset{\sim}{v}$ has magnitude 6 and direction $\frac{5\pi}{3}$ (1 mark)

The angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is $\frac{5\pi}{3} - \frac{5\pi}{4} = \frac{5\pi}{12}$

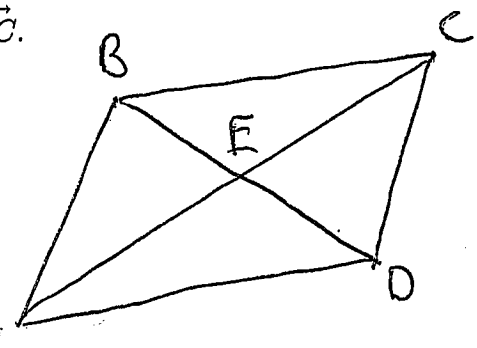
Therefore, $\underset{\sim}{u}.\underset{\sim}{v} = 8\times 6\times\cos\left(\frac{5\pi}{12}\right)$

$$\approx 12.42 \qquad (1 \text{ mark})$$

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15. Draw a parallelogram with vertices A, B, C, D such that $\overrightarrow{AD} = \overrightarrow{BC}$. Draw a line from B to D and a line from A to C, and let E denote the point of intersection of these two lines. Show that E is the midpoint of both lines by proving that

$$\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BD}, \quad \overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}.$$



(4 marks)

Since $\overrightarrow{BE}$ and $\overrightarrow{BD}$ are in the same direction,

$\overrightarrow{BE} = x\overrightarrow{BD}$ for some scalar x.

$\underset{\sim}{e} - \underset{\sim}{b} = x(\underset{\sim}{d} - \underset{\sim}{b})$

$\underset{\sim}{e} = (1-x)\underset{\sim}{b} + x\underset{\sim}{d}$ ①

(1 mark)

Since $\overrightarrow{AE}$ and $\overrightarrow{AC}$ are in the same direction

$\overrightarrow{AE} = y\overrightarrow{AC}$ for some scalar y,

$\underset{\sim}{e} - \underset{\sim}{a} = y(\underset{\sim}{c} - \underset{\sim}{a})$

$\underset{\sim}{e} = (1-y)\underset{\sim}{a} + y\underset{\sim}{c}$ ②

(1 mark)

Now $\overrightarrow{AD} = \overrightarrow{BC} \Rightarrow \underset{\sim}{d} - \underset{\sim}{a} = \underset{\sim}{c} - \underset{\sim}{b} \Rightarrow \underset{\sim}{c} = \underset{\sim}{b} + \underset{\sim}{d} - \underset{\sim}{a}$ ③

(1 mark)

Substituting ③ into ② gives

$\underset{\sim}{e} = (1-2y)\underset{\sim}{a} + y\underset{\sim}{b} + y\underset{\sim}{d}$. ④

Comparing ① and ④ we must have.

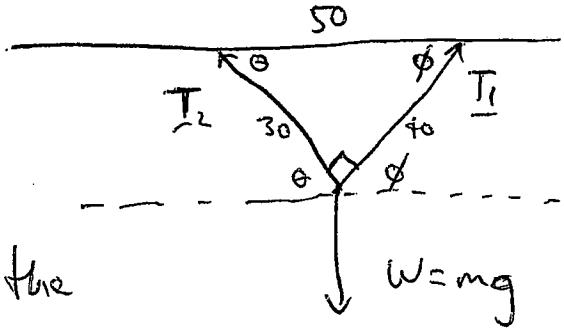
$1-2y=0$, $1-x=y$, $x=y \Rightarrow x=y=\frac{1}{2}$

$\therefore E$ is the midpoint as $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BD}$, $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$.

(1 mark).

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16. Two lengths of rope, one 30 cm long and the other 40 cm long, are attached to a ceiling a distance of 50 cm apart. The other ends of the rope are joined together and attached to a basket weighing 0.5 kg, which hangs freely. Determine the tension in each rope. (The acceleration due to gravity should be taken as $g = 9.8$ m/s^2.) (5 marks)



Let $\underline{T}_1, \underline{T}_2$ be the tensions in the ropes, and θ, ϕ the angles as shown on the diagram. Since $30^2 + 40^2 = 50^2$, the angle between the ropes is 90°. Now,

$\sin\phi = \frac{3}{5} \Rightarrow \phi = 36.9^\circ$, $\sin\theta = \frac{4}{5} \Rightarrow \theta = 53.1^\circ$. (1 mark)

$\underline{W} = -0.5 \times 9.8\,\underline{j} = -4.9\,\underline{j}$ ($\frac{1}{2}$ mark)

$\underline{T}_1 = T_1\cos\phi\,\underline{i} + T_1\sin\phi\,\underline{j} = \frac{4}{5}T_1\,\underline{i} + \frac{3}{5}T_1\,\underline{j}$ (1 mark)

$\underline{T}_2 = -T_2\cos\theta\,\underline{i} + T_2\sin\theta\,\underline{j} = -\frac{3}{5}T_2\,\underline{i} + \frac{4}{5}T_2\,\underline{j}$ (1 mark)

Since the mass is at rest, the total force is zero:

$\underline{W} + \underline{T}_1 + \underline{T}_2 = \underline{0}$ ($\frac{1}{2}$ mark)

$\therefore \left(\frac{4}{5}T_1 - \frac{3}{5}T_2\right)\underline{i} + \left(\frac{3}{5}T_1 + \frac{4}{5}T_2 - 4.9\right)\underline{j} = 0.$

$\Rightarrow T_1 = \frac{3}{4}T_2$ — (1) $\quad \frac{3}{5}T_1 + \frac{4}{5}T_2 = 4.9$ — (2)

Substituting (1) into (2) gives

$\frac{3}{5}\times\frac{3}{4}T_2 + \frac{4}{5}T_2 = 4.9 \Rightarrow \frac{25}{20}T_2 = 4.9 \Rightarrow T_2 = 3.92$

from (1) $T_1 = \frac{3}{4}\times 3.92 = 2.94$.

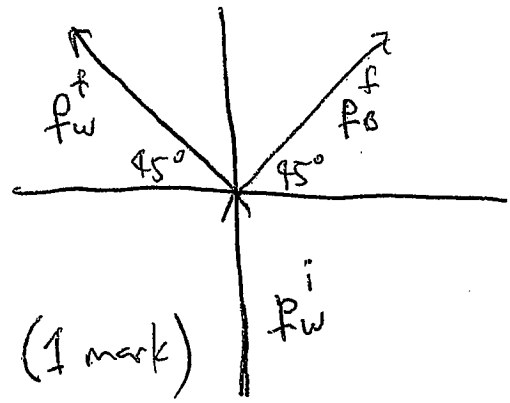
The tension in the 40 cm rope is 2.94 Newtons at 36.9 about the horizontal, and the tension in the other rope is 3.92 Newtons at 53.1° above the horizontal. (1 mark)

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MATH1050 — Mathematical Foundations
Mid Semester Examination, April 7-10, 2006 (continued)

17. On a snooker table (assumed to be frictionless), the white ball is travelling north at a speed of 1 m/s and the black ball is at rest. They collide, and after the collision the white ball rolls in the north-west direction while the black ball moves in the north-east direction at 0.6 m/s. If the mass of the white ball is 10 grams, what is the mass of the black ball in grams (to 2 decimal places). (5 marks)

Let X denote the speed of white ball after the collision, and let m_B denote the mass of the black ball.



For the white ball,

$$\underset{\sim}{p}_w^i = m_w \underset{\sim}{v}_w^i = 10 \times 1 \, \underset{\sim}{j} = 10 \, \underset{\sim}{j} \quad (1 \text{ mark})$$

$$\underset{\sim}{p}_w^f = m_w \underset{\sim}{v}_w^f = 10 \times \left(-X \cos 45^\circ \, \underset{\sim}{i} + X \sin 45^\circ \, \underset{\sim}{j} \right)$$

$$= -\frac{10X}{\sqrt{2}} \underset{\sim}{i} + \frac{10X}{\sqrt{2}} \underset{\sim}{j}. \quad (1 \text{ mark})$$

For the black ball, $\underset{\sim}{p}_B^i = 0$

$$\underset{\sim}{p}_B^f = m_B \underset{\sim}{v}_B^f = m_B \times \left(0.6 \cos 45^\circ \, \underset{\sim}{i} + 0.6 \sin 45^\circ \, \underset{\sim}{j} \right)$$

$$= \frac{0.6 m_B}{\sqrt{2}} \underset{\sim}{i} + \frac{0.6 m_B}{\sqrt{2}} \underset{\sim}{j}. \quad (1 \text{ mark})$$

Conservation of momentum is $\underset{\sim}{p}_w^i + \underset{\sim}{p}_B^i = \underset{\sim}{p}_w^f + \underset{\sim}{p}_B^f$ (1 mark)

$$\therefore \quad 10 \, \underset{\sim}{j} = -\frac{10X}{\sqrt{2}} \underset{\sim}{i} + \frac{10X}{\sqrt{2}} \underset{\sim}{j} + \frac{0.6 m_B}{\sqrt{2}} \underset{\sim}{i} + \frac{0.6 m_B}{\sqrt{2}} \underset{\sim}{j}.$$

$$\Rightarrow \quad \frac{0.6 m_B}{\sqrt{2}} - \frac{10X}{\sqrt{2}} = 0, \quad -(1), \qquad 10 = \frac{0.6 m_B}{\sqrt{2}} + \frac{10X}{\sqrt{2}} \quad -(2).$$

$$(1)+(2) \Rightarrow \quad \frac{2 \times 0.6 m_B}{\sqrt{2}} = 10 \Rightarrow m_B = \frac{10\sqrt{2}}{1.2} \approx 11.79 \text{ grams}$$

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The mass of the black ball is 11.79 grams. (1 mark

18. This question is about proving the Sine Law and the Cosine Law, with reference to the figure below.

(a) Find two expressions for the length d, one in terms of the angle A and one in terms of the angle B. (2 marks)

(b) From the results of (a), deduce that the Sine Law

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

holds. (1 mark)

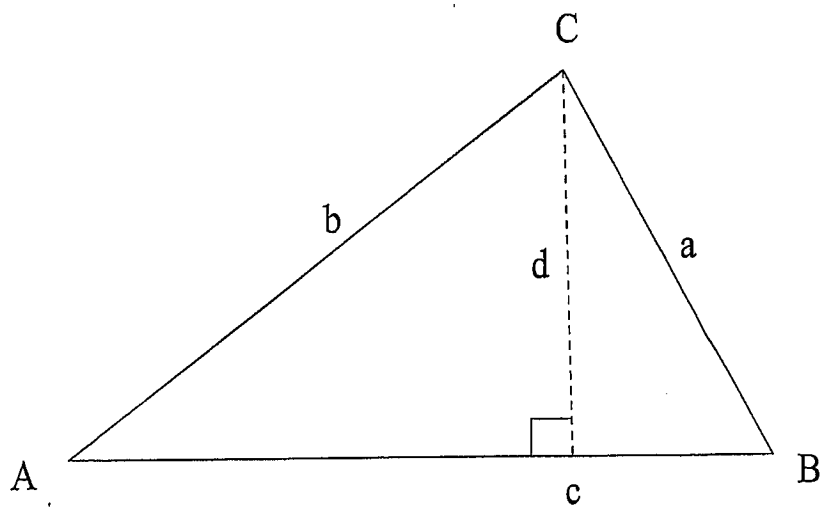
If A is at the origin

(c) The co-ordinates of the point C are $(b\cos A, b\sin A)$, while the co-ordinates of the point B are $(c, 0)$. In terms of these cordinates, calculated the magnitude of $\overrightarrow{BC}$. (3 marks)

(d) From the results of (c), deduce that the Cosine Law

$$a^2 = b^2 + c^2 - 2bc\cos A$$

holds. (1 mark)



(a) From the diagram,

$\sin A = \frac{d}{b} \Rightarrow d = b\sin A$ (1 mark)

$\sin B = \frac{d}{a} \Rightarrow d = a\sin B$ (1 mark)

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18. continued

(b). From (a), $b \sin A = a \sin B$

$$\Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} \quad (1 \text{ mark})$$

(c) $\vec{BC} = (b\cos A, b \sin A) - (c, 0)$

$$= (b\cos A - c, b \sin A) \quad (1 \text{ mark})$$

$$\|\vec{BC}\| = \sqrt{(b\cos A - c)^2 + (b\sin A)^2} \quad (1 \text{ mark})$$

$$= \sqrt{b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A}$$

$$= \sqrt{b^2 + c^2 - 2bc\cos A} \quad (1 \text{ mark})$$

(d) $a = \|\vec{BC}\|$

$$\therefore a^2 = \|\vec{BC}\|^2$$

$$= b^2 + c^2 - 2bc\cos A \quad (1 \text{ mark})$$

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19. Find the exact value of $\sin\left(\frac{17\pi}{12}\right)$ (where the angle is measured in radians). (3 marks)

$$\sin\left(\frac{17\pi}{12}\right) = \sin\left(\pi + \frac{5\pi}{12}\right)$$

$$= -\sin\left(\frac{5\pi}{12}\right) \qquad (1 \text{ mark})$$

$$= -\sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right)$$

$$= -\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= -\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \quad (1 \text{ mar}$$

$$= -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= -\frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right) \qquad (1 \text{ mark})$$

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20. Uschi and Hans are two MATH1050 students who decide to celebrate at The Red Room after finishing their midsemester exam. Uschi drinks 5 beers and 1 cocktail costing a total of \$41. Hans drinks 3 beers and 3 cocktails which costs him \$45. Using matrices, determine the price of a beer and the price of a cocktail. (4 marks)

Let B be the price of a beer, and C the price of a cocktail. Then

$$5B + C = 41$$
$$3A + 3C = 45$$

(1 mark)

As a matrix equation, this is

$$\begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} 41 \\ 45 \end{pmatrix}$$

(1 mark)

$$\det\begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix} = 5\times 3 - 3\times 1 = 15 - 3 = 12.$$

$$\begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix}^{-1} = \frac{1}{12}\begin{pmatrix} 3 & -1 \\ -3 & 5 \end{pmatrix}$$

(1 mark)

$$\begin{pmatrix} B \\ C \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 3 & -1 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 41 \\ 45 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 123 - 45 \\ -123 + 225 \end{pmatrix} = \begin{pmatrix} 6.5 \\ 8.5 \end{pmatrix}$$

A beer costs \$6.50 and a cocktail cost \$8.50

(1 mark).

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END OF EXAMINATION