MATH1050- Mathematical Foundations

PERUSAL TIME 10mins. During perusal, write on the blank paper provided
WRITING TIME 120 Minutes
EXAMINER Mr Michael Jennings

NO. OF PAGES (include title page and attachments) 17 Pages - Single-Sided

Exam Type: Closed Book - No materials permitted
Permitted Materials: Calculator - Yes - Non-programmable calculators only
Dictionary - No
Answer: On examination paper in spaces provided
Number of Questions: 24
Weighting/Marks: 50% / 100 marks
Special Instructions: Students must comply with the General Award Rules 1A.7 and 1A.8 which outline the responsibilities of students during an examination.

In Part B all working must be shown

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TOTAL

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM
PART A: For each of the following twelve multiple choice questions, enter the letter corresponding to the correct answer in the corresponding box. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. If $|x| + 2 < 4$ which of the following statements is true?
   (A) $-2 \leq x < 2$
   (B) $-2 < x < 2$
   (C) $-2 \leq x \leq 2$
   (D) none of the above
   Answer to Question 1: 

2. Which of the following statements is true, for all real numbers $a$ and $b$?
   (A) $|a + b| = |a| + |b|$
   (B) $|a| = -|a|$
   (C) $|ab| = |a| \times |b|$
   (D) none of the above
   Answer to Question 2: 

3. Which of the following intervals corresponds to the solution of the inequality $| - x | + 2 < 4$?
   (A) $[-2, 2)$
   (B) $(-2, 2)$
   (C) $(-\infty, -2) \cup (2, \infty)$
   (D) $(-\infty, -2] \cup [2, \infty)$
   Answer to Question 3: 

4. Let $A = (a_{ij})$ be an $n \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix, where $n$ and $p$ are distinct numbers. Which of the following statements is true?
   (A) $A + B$ is defined
   (B) $A \times B$ will be an $n \times n$ matrix
   (C) $2A$ is a $2n \times 2n$ matrix
   (D) $A + 0_{n \times n} = AI_{n \times n}$
   Answer to Question 4: 

Question 5 see next page.
5. If \( S = x^2 + 2x^4 + 3x^6 + 4x^8 \), which of the following statements is true?

(A) \( S = \sum_{i=1}^{4} 2ix^{2i} \)

(B) \( S = \sum_{i=1}^{4} 2ix^{4i} \)

(C) \( S = \sum_{i=0}^{3} (i + 1)x^{2i+2} \)

(D) \( S = \sum_{i=0}^{4} ix^{2i} \)

Answer to Question 5:  

6. If \( S = 4x^2 - 4x^4 + 4x^6 - 4x^8 \), which of the following statements is true?

(A) \( S = \sum_{i=1}^{4} (-1)^i ix^{2i} \)

(B) \( S = \sum_{i=1}^{4} (-1)^{i+1} ix^{2i} \)

(C) \( S = 4 \sum_{i=1}^{4} (-1)^{i+1} x^{2i} \)

(D) \( S = 4 \sum_{i=0}^{4} (-1)^{i+1} x^{2i} \)

Answer to Question 6:  

7. If \( y = e^{x^2} \sin(x^2) \) which of the following statements is true?

(A) \( \frac{dy}{dx} = 2xe^{x^2} \)

(B) \( \frac{dy}{dx} = 2xe^{x^2} (\sin(x^2) + \cos(x^2)) \)

(C) \( \frac{dy}{dx} = e^{x^2} (\sin(x^2) + \cos(x^2)) \)

(D) \( \frac{dy}{dx} = 2e^{x^2} (\sin(x^2) + \cos(x^2)) \)

Answer to Question 7:  

8. If \( y = 6x^2 + 8x^3 + 10x^4 + 12x^5 \) which of the following statements is true?

(A) \( \int y \, dx = 12x + 24x^2 + 40x^3 + 60x^4 + C \)

(B) \( \int y \, dx = 2(x^3 + x^4 + x^5 + x^6) + C \)

(C) \( \int y \, dx = x^3 + x^4 + x^5 + x^6 + C \)

(D) \( \int y \, dx = 2(6x + 12x^2 + 20x^3 + 30x^4) + C \)  

Answer to Question 8:  

Question 9 see next page.
9. If \( y = 3 \sin x \) which of the following statements is true?

(A) \( \int y \, dx = -3 \cos x + C \)

(B) \( \int y \, dx = -3 \cos x^2 + C \)

(C) \( \int y \, dx = -3 \cos (x^2) + C \)

(D) \( \int y \, dx = 3x \cos x + C \)

Answer to Question 9:

10. Let \( w = 2 - 3i \) and \( z = 1 + 5i \). Which of the following statements is true?

(A) \( 3w - z = 5 - 14i \)

(B) \( 3w - z = 1 - 9i \)

(C) \( 3w - z = 7 - 6i \)

(D) none of the above

Answer to Question 10:

11. Let \( w = 2 - 3i \). Which of the following statements is true?

(A) \( |w|^2 = \sqrt{5} \)

(B) \( |w|^2 = -1 \)

(C) \( |w|^2 = \sqrt{13} \)

(D) \( |w|^2 = 13 \)

Answer to Question 11:

12. Let \( u = 5i \) and \( z = 1 + 2i \). Which of the following statements is true?

(A) \( \frac{u}{z} = 1 + 5i \)

(B) \( \frac{u}{z} = 5 - i \)

(C) \( \frac{u}{z} = 2 + i \)

(D) \( \frac{u}{z} = -2 - i \)

Answer to Question 12:

END PART A

Question 13 see next page.
PART B: SHOW ALL WORKING

13. Austyn and Bernadette have found that they are becoming addicted to Mars Bars and their weekly consumption is increasing. The following sequences represent the number of Mars Bars eaten by Austyn and Bernadette to the thirteenth week of semester. Note that in some weeks they were generous, sharing the Mars Bars with friends, hence the total includes a fraction of a Mars Bar.

Austyn: \( a_1 = 1, a_n = \frac{4a_{n-1}}{3}, \) for \( n = 2, \ldots, 13. \)

Bernadette: \( b_1 = 3, b_n = b_{n-1} + \frac{1}{3}, \) for \( n = 2, \ldots, 13. \)

a) (2 marks) Find closed form representations for each of the two sequences.

b) (2 marks) Calculate the value of each of the terms \( a_{13} \) and \( b_{13}. \)

c) (4 marks) Using the formula for either an arithmetic series or a geometric series, calculate how many Mars Bars Austyn eats in 13 weeks, and calculate the total number of Mars Bars eaten by Bernadette in 13 weeks.
14. Let \( z = (1 + i\sqrt{3})^{-2} \) be a complex number.

a) (4 marks) Find the real and imaginary parts of \( z \).

b) (2 marks) Find \(|z|\).

c) (2 marks) Write \( z \) in modulus and argument form.

d) (2 marks) Calculate \( z^4 \).
15. Find the following integrals

a) (4 marks) $\int_{1}^{e} \frac{\sin(ln \ x)}{x} \, dx$

b) (5 marks) $\int x e^{-x^2} \, dx$
16. Let \( f(x) = 15 + 2x^2 - x^4 \).

   a) (4 marks) Find the local maxima and local minima of \( f \).

   b) (3 marks) Sketch the graph of \( f \). Make sure you clearly identify where the graph intercepts the axes, and the value of the function at the critical points.
17. (5 marks) Julie and Johan enjoy the creamy caramel nougat of a Mars Bar, but not the appearance of the chocolate coating. Consequently they would like the shape of a Mars Bar redesigned to minimize the surface area. So assume that a 60gm Mars Bar has a rectangular box shape as shown in the diagram. Also assume that the volume of the 60 gm Mars Bar is 37.5 cm³ and the length of the Mars Bar is fixed at 10 cm. Calculate the value of the width, $W$, and the height, $H$, so that the surface area of the 60gm Mars Bar is minimized.

A 60gm Mars Bar with volume 37.5 cm³
18.  a) (4 marks) Find the values of \( x \) such that
\[
\frac{|x + 1|}{2} < 1.
\]

b) (5 marks) Find the values of \( x \) such that
\[
\frac{1}{x - 1} + \frac{1}{x + 1} \leq 0.
\]
19. Let \( f(x) = 2 \sin x \).

   a) (4 marks) Evaluate the Riemann sum \( A_L = \sum_{i=1}^{n} f(x_{i-1}) \Delta x \) for \( f(x) \) with \( 0 \leq x \leq \pi \), and taking six subintervals.

   b) (2 marks) Evaluate the definite integral \( \int_{0}^{\pi} f(x) \, dx \).

   c) (1 mark) Explain the difference between the answer for Part a) and Part b).
20. (5 marks) Use Mathematical Induction to prove

\[
\sum_{i=1}^{n} (2i - 1)^2 = \frac{n(4n^2 - 1)}{3}
\]

for integers \( n \geq 1 \).
21. (4 marks) A 60gm Mars Bar is resting on a desk inclined at an angle of 30° to the horizontal. The only forces acting on the Mars Bar are weight, the normal reaction to the desk and friction which keeps the Mars Bar at rest. Determine the magnitude of each of the three forces. (Give your answers to two decimal place accuracy.)
22. (5 marks) Find the equation of the tangent line to the graph of

\[ y^4 + 4y = 4x^3 + 1 \]

at the point \( P = (1,1) \).
23. Let \( f(x) = \frac{1}{2x-4} \) and \( g(x) = x^2 + 2 \).
   
a) (4 marks) Write down the domain and range for the function \( f \) and \( g \).

b) (3 marks) Is \( g(x) \) a one-to-one function. If so prove it, if not give a counter example.

c) (4 marks) Write down an expression for \( f \circ g \) and \( g \circ f \). Simplify if possible.
In each case state the domain of the composite function.
24. Let \[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & -\cos x \end{pmatrix}. \]

a) (2 marks) Prove \( \det(A) = -\cos(2x) \).

b) (1 mark) For what values of \( x \) in the interval \([0, 2\pi]\) is the matrix \( A \) singular?

c) (5 marks) Show

\[ A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\sin(2x) \\ 0 & -\sin(2x) & 1 \end{pmatrix}. \]
\[
\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi \\
\sin(\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi \\
\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \\
\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi \\
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \\
\sin 2\theta = 2 \sin \theta \cos \theta
\]

For an arithmetic sequence of the form \( \{a_j\}_{j=1}^n \), where \( a_j = a + (j - 1)d \),

\[
S_n = \sum_{i=1}^{n} a_i = \frac{n}{2} (2a + (n - 1)d).
\]

For a geometric sequence of the form \( \{a_j\}_{j=1}^n \), where \( a_j = ar^{j-1} \),

\[
S_n = \sum_{i=1}^{n} a_i = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}.
\]