Exam Type: Closed Book
Calculator: Yes - Any
Dictionary: No
Permitted Materials:

SPECIAL INSTRUCTIONS TO STUDENTS:
There are 100 marks on the paper. All questions carry the indicated number of marks, and students should aim to complete all questions. Credit will be given only for work written on this examination script. Use the back pages if the space provided is insufficient. Page 17 is a reference page; students may detach that page. Students are allowed any calculator, but the memory of any programmable calculator must be reset immediately prior to the examination.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

General Award Rules 1A.7 and 1A.8

1A.7 Responsibilities of students
1A.7.1 A student must comply with the examination instructions and directions given by an examination supervisor.
1A.7.2 A student may not enter an examination room without the permission of the examination supervisor, or after the first 10 minutes of examination working time.
1A.7.3 A student must not leave the examination room without the permission of the examination supervisor.
1A.7.4 Permission will not be granted under GAR 1A.7.3 during —
(a) the first 10 minutes of examination working time; and
(b) the final 10 minutes.
1A.7.5 (a) A student must bring into the examination some identification in the form of a current student card or other photographic identification.
(b) The identification must be displayed throughout the examination.
(c) Before the start of the examination, a student without identification must sign a declaration in a form set by the secretary and registrar.
(d) A student who does an examination without identification must produce identification at a location specified in writing by the secretary and registrar either generally or for that student.
(e) The university may withhold the results for an examination for a student who did not have identification at the examination until the student has produced identification under GAR 1A.7.5(d).
1A.7.6 Unless addressing a question to the examiner or examination supervisor, a student must not communicate in any way with another person during the examination.
1A.7.7 A student may bring unauthorised material into the examination room only if the material —
(a) is brought in with the permission of the examiner or examination supervisor;
(b) is left with the examination supervisor immediately on entering the examination room.
1A.7.8 A student may remove examination books, scripts or material provided to the student during the examination only with the permission of the examination supervisor.

1A.8 Examination supervisors
1A.8.1 The examination supervisor may —
(a) inspect any material brought into the examination room by a student; and
(b) confiscate any material which the examination supervisor reasonably suspects to be or to contain unauthorised material.
1A.8.2 If the examination supervisor reasonably believes that a student's behaviour may distract or disturb other students, the examination supervisor may direct the student to leave the examination room.
1. Determine all values of \( k \) such that the following system of equations has a unique solution.

\[
\begin{align*}
    x + 2y - 3z &= 4 \\
    3x - y + 5z &= 12 \\
    4x + y + kz &= 5
\end{align*}
\]

If \[
\begin{vmatrix}
    1 & 2 & -3 \\
    3 & -1 & 5 \\
    4 & 1 & k
\end{vmatrix} \neq 0,
\]
then the system has a unique solution.

\[
\begin{vmatrix}
    -1 & 5 \\
    4 & k \\
    1 & -3
\end{vmatrix} + 3
\]

\[
= 1 (-k - 5) - 2 (3k - 20) - 3 (1 + 4)
\]

\[
= -k - 5 - 6k + 40 - 9 - 12
\]

\[
= -7k - 14
\]

So \( 7k \neq -14 \) \( \Rightarrow k \neq -2 \). So any value except \( k = -2 \) will give a unique solution.

Question 2 see next page.

TURN OVER.
Three potential astronauts, Naomi, Dean and Jay, are trying out the zero gravity room at the space training centre.

The mass of each person in their spacesuit is as follows: Naomi 65kg, Dean 75kg and Jay 72kg. They push off from the walls of the room in the directions shown with the following velocities: Naomi 2.5 m/s, Dean 2.8 m/s and Jay 3.0 m/s. When they collide, they hold onto each other and move off as one object after the collision. Determine their speed and direction after the collision.

Round the speed to one decimal place accuracy and give the direction as a bearing rounded to the nearest degree.

(8 marks)

\[
\text{Mom. before} \\
\text{Horiz. } (65 \times 2.5 \times \cos 53) + (-72) + (-75 \times 2.8 \times \cos 45) \\
\approx -266.7 \text{ Ns}
\]

\[
\text{Vert} \quad 65 \times 2.5 \times \cos 53 + 0 - 75 \times 2.8 \times \sin 45 \\
\approx -18.7
\]

Let Mom. after

\[
= x.
\]

Question 3 see next page. 18.7

\[
0 \approx 4.0^\circ \quad \Rightarrow \text{Bearing} \ 184.0^\circ \text{ (from N + E)}
\]

\[
2 \approx \sqrt{18.7^2 + 266.7^2} \approx 227.5 \text{ NS}
\]

\[
\text{Speed is} \quad \approx \frac{227.5}{2.12} \\
\approx 107 \text{ m/s}
\]
3. Use mathematical induction to prove that for all integers \( n \geq 1, \)
\[
\sum_{i=1}^{n} (2i - 5) = n^2 - 4n.
\]

Let \( P(n) \) be the claim that
\[
\sum_{i=1}^{n} (2i - 5) = n^2 - 4n, \text{ for all } n \geq 1.
\]

\( P(1) \) holds

\[
\begin{align*}
\text{LHS } n=1 & : \sum_{i=1}^{1} (2i - 5) = -3 \\
\text{RHS } n=1 & : 1^2 - 4 \times 1 = -3
\end{align*}
\]

Assume \( P(k) \) holds
\[
\sum_{i=1}^{k} (2i - 5) = k^2 - 4k
\]

Inductive assumption

\( P(k+1) \) also holds
\[
\begin{align*}
\text{LHS } n=k+1 & : \sum_{i=1}^{k+1} (2i - 5) = \sum_{i=1}^{k} (2i - 5) + (2(k+1) - 5) \\
& = k^2 - 4k + 2(k+1) - 5 \\
& = k^2 - 2k - 3
\end{align*}
\]

\( \text{RHS } n=k+1 \)

\[
(k+1)^2 - 4(k+1) = k^2 + 2k + 1 - 4k - 4 = k^2 - 2k - 3 = \text{LHS}
\]

\[
\sum_{i=1}^{n} (2i - 5) = n^2 - 4n \text{ by mathematical induction for all } n \geq 1.
\]
4. Determine the sum of the terms of the following sequence.

\[ a_n = 3n + 1 \quad (n = 1, 2, 3, \ldots, 100) \]

\[ a = 4 \quad d = 3 \]

\[ S = \sum_{i=1}^{100} a_n = \frac{100}{2} (2 \times 4 + 99 \times 3) \]

\[ = 50 (8 + 297) \]

\[ = 15250 \]

(3 marks)

5. Determine which term of the following sequence is the first term to exceed 100,000.

\[ b_{n+1} = 2b_n \quad (n = 1, 2, \ldots) \quad \text{where } b_1 = 3 \]

(4 marks)

\[ r = 2 \quad b_1 = 3 \]

\[ b_n = b_1 \cdot r^{n-1} \]

\[ 100,000 < 3 \cdot 2^{n-1} \]

\[ 2^{n-1} > \frac{100,000}{3} \]

\[ n-1 > \ln \frac{100,000}{3} \]

\[ n > \ln \frac{100,000}{3} \]

\[ n > 16.02 \]

The 17th term will be the first term to exceed 100,000.
6. Solve the following inequality. Write your solution in interval notation and illustrate it on a number line.

\[ \frac{|x - 4|}{2x + 3} \leq 1 \]

(Hint: You may want to consider three cases, based on where the numerator and denominator of the fraction change sign.)

\[
\frac{2x+3}{x-4} \leq \frac{2x+3}{2x+3} \\
\frac{x-4}{2x+3} \leq 1 \\
\frac{x-4}{2x+3} \leq 2x+3 \]

\[
x - 4 \leq 2x + 3 \quad \text{or} \quad -(x - 4) \leq 2x + 3 \\
x \geq -7 \\
-x + 4 \leq 2x + 3 \\
1 \leq 3x \\
x \geq \frac{1}{3}
\]

\[
\frac{2x+3}{x-4} \geq \frac{2x+3}{2x+3} \\
\frac{x-4}{2x+3} \geq 1 \\
\frac{x-4}{2x+3} \geq 2x+3 \]

\[
x - 4 \geq 2x + 3 \quad \text{or} \quad -(x - 4) \geq 2x + 3 \\
x \leq -7 \\
-x + 4 \geq 2x + 3 \\
3x \leq 1 \\
x \leq \frac{1}{3}
\]

Question 7 see next page.

\[ \text{Ans: } (-\infty, -7] \cup \left[ \frac{1}{3}, \infty \right) \]
7. (a) Write the complex number $1 - \sqrt{3}i$ in polar form using the principal argument.

Let $z = 1 - \sqrt{3}i$.

$$|z| = \sqrt{1^2 + \sqrt{3}^2} = 2$$

$$\tan \theta = -\sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

So $z = 2 \operatorname{cis} \left( -\frac{\pi}{3} \right)$

(b) Determine $(1 - \sqrt{3}i)^8$ and give your answer in Cartesian form.

$$z = 2 \operatorname{cis} \left( -\frac{\pi}{3} \right)$$

$$z^8 = 2^8 \operatorname{cis} \left( 8 \times -\frac{\pi}{3} \right) = 256 \operatorname{cis} \left( -\frac{8\pi}{3} \right)$$

$$= 256 \cos \left( -\frac{8\pi}{3} \right) + i256 \sin \left( -\frac{8\pi}{3} \right)$$

$$= -128 - 128\sqrt{3}i$$
8. One solution of $3z^3 - 11z^2 + 11z + 5 = 0$ is $z = 2 - i$. Write $3z^3 - 11z^2 + 11z + 5$ as the product of linear factors and hence find all solutions, over $\mathbb{C}$, of

$$3z^3 - 11z^2 + 11z + 5 = 0.$$ (7 marks)
9. Let $f(x) = \ln(x^2)$ and let $g(x) = \sqrt{x - 1}$.

(a) State the domain and range of $f$. 
\[ D : \mathbb{R} \setminus \{0\} \]
\[ R : \mathbb{R} \]

(b) State the domain and range of $g$. 
\[ D : [1, \infty) \]
\[ R : [0, \infty) \]

(c) Determine $(f \circ g)(x)$ and state its domain and range.
\[ f(g(x)) = \ln \left( \sqrt{x - 1} \right)^2 \]
\[ = \ln (x - 1) \]
\[ D : (1, \infty) \]
\[ R : (-\infty, \infty) \]

(d) Is $f$ a one-to-one function? Explain your answer. 
\[ \text{No, } \because f(1) = f(-1) \]

Question 10 see next page.
10. Let \( f(x) = \sqrt{x - 4} \).

(a) Use the definition of the derivative to calculate \( f'(8) \).

\[
\begin{align*}
\lim_{h \to 0} & \frac{f(8 + h) - f(8)}{h} \\
= & \lim_{h \to 0} \frac{\sqrt{8 + h - 4} - \sqrt{8 - 4}}{h} \\
= & \lim_{h \to 0} \frac{\sqrt{8 + h} - \sqrt{4}}{\sqrt{8 + h} + \sqrt{4}} \\
= & \lim_{h \to 0} \frac{\frac{8 + h - 4}{\sqrt{8 + h} + \sqrt{4}}}{h} \\
= & \lim_{h \to 0} \frac{4}{(\sqrt{8 + h} + \sqrt{4})h} \\
= & \frac{1}{\sqrt{8} + \sqrt{4}} \quad \text{\( \therefore f'(8) = \frac{1}{4} \)}
\end{align*}
\]

(b) Determine the equation of the tangent line to \( f \) at \( x = 8 \).

\[
y = m(x - 8) + C \\
m = \frac{1}{4} \quad \Rightarrow \quad y = \frac{1}{4} x + C
\]

\[
\begin{align*}
x = 8 & \quad \Rightarrow \quad y = 2 \\
\Rightarrow & \quad y = \frac{1}{4} (8) \\
& \quad \Rightarrow \quad y = 2
\end{align*}
\]

\[
\therefore C = 0
\]

\[
\therefore \quad y = \frac{1}{4} x
\]

Question 11 see next page.
11. Determine the following limit.

\[
\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \frac{(x+1)(x-3)}{x-3} = \lim_{x \to 3} (2x+1) = 7
\]

(3 marks)

12. Find the derivative of each of the following functions.

(a) \(f(x) = \ln \sqrt{4x - 1}\)  

\[
\text{Let } u = \sqrt{4x - 1} = (4x - 1)^{1/2}, \text{ then } y = \ln u
\]

\[
\frac{du}{dx} = 2(4x - 1)^{-1/2}, \quad \frac{dy}{du} = \frac{1}{u}
\]

\[
\Rightarrow \quad f'(x) = \frac{2}{\sqrt{4x - 1} \cdot \sqrt{4x - 1}} = \frac{2}{4x - 1}
\]

(2 marks)

(b) \(g(a) = \frac{\sin(a^3)}{5a}\)

\[
\text{Let } u = \sin(a^3), \quad v = 5a
\]

\[
u' = 3a^2 \cos(a^3), \quad v' = 5
\]

\[
g'(a) = \frac{15a^3 \cos(a^3) - 5\sin(a^3)}{25a^2}
\]

(2 marks)

Question 13 see next page.  

TURN OVER
13. A cable is to be laid from the island lighthouse \( L \) to a point \( B \) on the shore. The lighthouse lies 5 km directly north of a point \( A \) and the point \( B \) lies 10 km directly east of \( A \). The cable (dashed line below) will be laid through the water from \( L \) to a point \( C \) on the shore between \( A \) and \( B \) and from there to \( B \) above ground along the shore. The part of the cable from \( L \) to \( C \) costs $5000 per km and the part of the cable from \( C \) to \( B \) costs $3000 per km.

\[
\begin{array}{c}
A \quad x \quad C \quad \quad B
\end{array}
\]

(a) Let \( x \) be the distance from \( A \) to \( C \). Determine a formula for the cost \( C \) of the cable in terms of the variable \( x \).

\[
LC = \sqrt{x^2 + 25}
\]

\[
\therefore \quad C_{AC} = \sqrt{x^2 + 25} \cdot 5000 + (10-x) \cdot 3000
\]

\[
= 5000 \sqrt{x^2 + 25} - 3000x + 30000.
\]

(b) Show that 
\[
\frac{dC}{dx} = \frac{5000x}{\sqrt{25 + x^2}} - 3000.
\]

Let \( \omega = C(x) \)

\[
\therefore C(x) = 5000 \left(\frac{x^2 + 25}{125}\right)^{\frac{1}{2}} - 3000x + 30000
\]

\[
\equiv C'(x) = \frac{2x \cdot 5000}{2 \sqrt{x^2 + 25}} - 3000
\]

\[
= \frac{5000x}{\sqrt{x^2 + 25}} - 3000.
\]

Question 13 continues next page.
13. (continued) Recall that \( \frac{dC}{dx} = \frac{5000x}{\sqrt{25 + x^2}} - 3000. \)

(c) Solve \( \frac{dC}{dx} = 0. \)

(Hint: Isolate \( \sqrt{25 + x^2} \) and then square both sides of your equation.)

\[
3000 = \frac{5000x}{\sqrt{25 + x^2}}
\]

\[
3000 \sqrt{25 + x^2} = 5000x
\]

\[
9(25 + x^2) = 25x^2
\]

\[
16x^2 = 225
\]

\[
x^2 = \frac{225}{16}
\]

\[
x = \pm 3.75
\]

(d) Determine where \( C \) should lie so as to minimise the cost of the cable, and determine the minimum cost of the cable.

\[
C'(x) = \frac{5000x}{\sqrt{25 + x^2}} - 3000
\]

\[
C'(4) = +ve
\]

\[
C'(3\frac{3}{4}) = -ve
\]

Minimum when \( x = 3\frac{3}{4} \)

\[
= 5000 \sqrt{(3\frac{3}{4})^2 + 25} - 3000 \cdot 3\frac{3}{4}
\]

\[
= \$ 50000
\]

Question 14 see next page.
14. Evaluate the following integrals.

(a) \( \int \frac{15x^2 + 3}{5x^3 + 3x + 2} \, dx \) \\
\hspace{1cm} \text{Let } u = 5x^3 + 3x + 2 \hspace{1cm} \text{\( \int \frac{du}{u} = \ln|u| + C \)} \\
\hspace{1cm} \Rightarrow \int \frac{15x^2 + 3}{u} \, dx = \ln|u| + C \hspace{1cm} = \ln|5x^3 + 3x + 2| + C \hspace{1cm} (2 \text{ marks})

(b) \( \int e^{x^2 + e^x} \, dx \) \\
\hspace{1cm} \text{Let } u = x^2 + e^x \hspace{1cm} \Rightarrow \int u^{\frac{1}{2}} \, du = \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3} (1 + e^x)^{\frac{3}{2}} + C \hspace{1cm} (2 \text{ marks})

(c) \( \int_0^\frac{\pi}{2} (2 + \sin(x))^3 \cos(x) \, dx \) \\
\hspace{1cm} \text{Let } u = 2 + \sin x \hspace{1cm} \Rightarrow \int u^3 \, du = \left[ \frac{u^4}{3} \right]_0^{\tan x} \hspace{1cm} = \left[ \frac{(2 + \sin x)^4}{4} \right]_0^{\frac{\pi}{2}} \hspace{1cm} (3 \text{ marks})

\[ \int_0^{\frac{\pi}{2}} (2 + \sin x)^3 \cos x \, dx = \frac{\pi^3}{8} - \frac{2^3}{8} \]

Question 15 see next page. 

\[ \]
15. Let \( f(x) = x^2 - 2x + 3 \). Part of the graph of \( f \) is shown below. Evaluate the Riemann sum for \( f(x) \), with \(-1 \leq x \leq 2\), having 30 subintervals and taking the sample points to be the right endpoints of the subintervals.

(Hint: You might like to use the useful series from the reference page.)

\[
\Delta x = \frac{3}{30} = \frac{1}{10}.
\]

\[
A_k = \sum_{i=1}^{n} f(x_i) \cdot \Delta x.
\]

Similar to Q4.2.1.

of lecture notes.
16. Let \( f(x) = e^x - 1 \). The following diagram shows part of the graph of \( f \). Use definite integrals to determine the total area of the two regions bounded by the graph of \( f \), the \( x \)-axis, and the lines \( x = -2 \) and \( x = 2 \).

\[
\text{Area} = \left| \int_{-1}^{0} f(x) \, dx \right| + \int_{0}^{2} f(x) \, dx
\]

\[
= \left| \int_{-1}^{0} (e^x - 1) \, dx \right| + \int_{0}^{2} (e^x - 1) \, dx
\]

\[
= \left[ \frac{e^x}{x} - x \right]_{-1}^{0} + \left[ e^x - x \right]_{0}^{2}
\]

\[
= \left| e^0 - (e^{-1} + 2) \right| + e^2 - 2 - e^0
\]

\[
= \left| 1 - e^{-2} - 2 \right| + e^2 - 3
\]

\[
= \left| -1 - e^{-2} \right| + e^2 - 3
\]

\[
= e^{-2} + 1 + e^2 - 3
\]

\[
= e^2 + e^{-2} - 2.
\]
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$. If $ad - bc \neq 0$ then $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $||v|| = \sqrt{v_1^2 + v_2^2}$.

Momentum = Mass $\times$ Velocity.

Cosine Rule: $a^2 = b^2 + c^2 - 2bc \cos A$

Sine Rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic sequence with first term $a$ and common difference $d$.

Closed Form: $a_n = a + (n - 1)d$  $(n = 1, 2, \ldots)$  Partial sum: $S_n = \frac{n}{2}(2a + (n - 1)d)$

Recursive def: $a_{n+1} = a_n + d$  $(n = 1, 2, \ldots)$ and $a_1 = a$

Let $\{a_n\}_{n=1}^{\infty}$ be a geometric sequence with first term $a$ and common ratio $r$.

Closed form: $a_n = ar^{n-1}$  $(n = 1, 2, \ldots)$

Recursive def: $a_{n+1} = a_n \times r$  $(n = 1, 2, \ldots)$ and $a_1 = a$

Partial sum: $S_n = \frac{a(r^n - 1)}{r - 1}$  Infinite series: $\sum_{n=1}^{\infty} a_n = \frac{a}{1 - r}$ (if $|r| < 1$).

If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. If $z = r \cos \theta$ then $z^n = r^n \cos(n \theta)$.

If $z^n = r \cos \theta$ then $z = r^{\frac{1}{n}} \cos \left(\frac{\theta + 2k\pi}{n}\right)$ for $k \in \mathbb{Z}$.

Factorisation: $x^2 - y^2 = (x - y)(x + y)$  $x^2 + 2xy + y^2 = (x + y)^2$  $x^2 - 2xy + y^2 = (x - y)^2$

$a^2 + 2cx + c = (a_1x + c_1)(a_2x + c_2)$ where $a = a_1a_2$, $c = c_1c_2$ and $b = a_1c_2 + a_2c_1$

$f(x) = e^x$ has domain $\mathbb{R}$ and range $(0, \infty)$.

The derivative of a function $f$ at the number $a$ is $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.

Differentiation rules:

$(fg)' = fg' + gf'$

$\frac{f'}{g} = \frac{gf' - fg'}{g^2}$

$\frac{f}{g} = f \circ g)'(x) = f'(g(x))g'(x)$

$\frac{d}{dx}(\sin x) = \cos x$  $\frac{d}{dx}(\cos x) = -\sin x$  $\frac{d}{dx}(\tan x) = \sec^2 x$  $\frac{d}{dx}(e^x) = e^x$  $\frac{d}{dx}(\ln x) = \frac{1}{x}$

If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then $\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$.

Useful series: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{n(n+1)}{2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{n(n+1)(2n+1)}{6}$.