### MATH1050 MATHEMATICAL FOUNDATIONS

**PERUSAL TIME**
10 mins. During perusal, write on the blank paper provided

**WRITING TIME**
2:00 Hours

**EXAMINER**
MR MICHAEL JENNINGS

This examination paper has 16 pages (include title page and attachments) and is printed on Single-Sided

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**THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM**

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<tr>
<th>Exam Type:</th>
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<tr>
<td>Permitted Materials:</td>
<td>Calculator - Yes - Non-programmable calculators only</td>
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<td>Dictionary - Yes - Unmarked paper Bilingual Dictionary only</td>
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<td>- No electronic aids are permitted (e.g. laptops, phones)</td>
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<td>Answer: (Where students should write answers)</td>
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<td>Total Number of Questions: (for the whole examination)</td>
<td>24</td>
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Students must comply with the General Award Rules 1A.5 and 1A.7 which outline the responsibilities of students during an examination.
PART A: For each of the following twelve multiple choice questions, enter the letter corresponding to the correct answer in the corresponding box. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. Let $A = (a_{ij})$ be an $p \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix, where $n$ and $p$ are distinct numbers. Which of the following statements is true?

(A) $A + B$ is undefined
(B) $A \times B$ will be an $n \times n$ matrix
(C) $2A$ is a $2p \times 2n$ matrix
(D) $B + A$ is defined

Answer to Question 1: __A__

2. If $S = (x^3 - 1) + (x^4 - 2) + (x^5 - 3)$, which of the following statements is true?

(A) $S = \sum_{i=3}^{5}(x^i - i)$
(B) $S = \sum_{i=1}^{3}(x^i - i)$
(C) $S = \sum_{i=1}^{3}(x^{i+2} - i)$
(D) $S = \sum_{i=3}^{5}(x^{2i-3} - i + 2)$

Answer to Question 2: __C__

3. If $S = \sum_{i=0}^{3}(-1)^i(x+1)^{2i}$, which of the following statements is true?

(A) $S = -(x+1)^2 + (x+1)^4 - (x+1)^6$
(B) $S = 1 - (x+1)^2 + (x+1)^4 - (x+1)^6$
(C) $S = -1 - (x+1)^2 - (x+1)^4 - (x+1)^6$
(D) $S = 1 - (x+1)^2 + (x+1)^4 - (x+1)^6$

Answer to Question 3: __B__

4. If $f(x) = 3x - 2$, which of the following is the inverse function?

(A) $f^{-1}(x) = 3y - 2$
(B) $f^{-1}(x) = -3x + 2$
(C) $f^{-1}(x) = \frac{1}{3}x + \frac{2}{3}$
(D) $f^{-1}(x) = \frac{1}{3}x - \frac{1}{2}$

Answer to Question 4: __C__

Question 5 see next page.

TURN OVER
5. \( \log_{25} 5 \) equals
   
   (A) 5
   (B) 125
   (C) \( \frac{1}{2} \)
   (D) 2

   Answer to Question 5: \( \boxed{C} \)

6. Which of the following intervals corresponds to the solution of the inequality \(| -x | + 3 < 6\)?
   
   (A) \([-3,3]\)
   (B) \((-3,3)\)
   (C) \((-\infty,-3) \cup (3,\infty)\)
   (D) \((-\infty,-3) \cup [3,\infty)\)

   Answer to Question 6: \( \boxed{B} \)

7. Which of the following statements is true, for all real numbers \( a \) and \( b \)?
   
   (A) \( |a| \div |b| = |a| - |b| \)
   (B) \( |a| = -|a| \)
   (C) \(-|ab| = |a| \times -|b| \)
   (D) \( |a + b| = |a| + |b| \)

   Answer to Question 7: \( \boxed{C} \)

8. If \( y = x^2 \sin x \), which of the following statements is true?
   
   (A) \( \frac{dy}{dx} = x^2 \cos x + 2x \sin x \)
   (B) \( \frac{dy}{dx} = 2x \cos x + x^2 \sin x \)
   (C) \( \frac{dy}{dx} = 2x \sin x \)
   (D) \( \frac{dy}{dx} = 2x \cos x \)

   Answer to Question 8: \( \boxed{A} \)

Question 9 see next page.
9. If \( y = 6x^2 - 5 + 2e^x \) which of the following statements is true?

(A) \( \int y \, dx = 2x^3 - 5x + 2e^x + C \)

(B) \( \int y \, dx = 2x^3 - 5x + e^{2x} + C \)

(C) \( \int y \, dx = 2x^3 - 5 + e^{2x} + C \)

(D) \( \int y \, dx = 2x^3 - 5x + \frac{e^{2x}}{2} + C \)

Answer to Question 9: [Blank]

10. Let \( z = 1 - 4i \). Which of the following statements is true?

(A) \( |z| = 3 \)

(B) \( |z| = -15 \)

(C) \( |z| = \sqrt{17} \)

(D) \( |z| = 17 \)

Answer to Question 10: [Blank]

11. Let \( w = 2 - 3i \) and \( z = 1 - 4i \). Which of the following statements is true?

(A) \( 2w - z = 3 - 10i \)

(B) \( 2w - z = 3 - 2i \)

(C) \( 2w - z = 5 - 10i \)

(D) \( 2w - z = 3 - 7i \)

Answer to Question 11: [Blank]

12. Let \( u = -10i \) and \( z = 1 - 2i \). Which of the following statements is true?

(A) \( \frac{u}{z} = 1 + 5i \)

(B) \( \frac{u}{z} = -4 + 2i \)

(C) \( \frac{u}{z} = -2i \)

(D) \( \frac{u}{z} = 4 - 2i \)

Answer to Question 12: [Blank]
PART B: SHOW ALL WORKING

13. Let \( f(x) = -x^3 + 9x^2 - 24x - 3 \).
   
   a) (5 marks) Find the local maxima and local minima of \( f \).

   \[
   f'(x) = -3x^2 + 18x - 24
   \]

   Critical points when \( f'(x) = 0 \)

   \[
   \Rightarrow -3x^2 + 18x - 24 = 0
   \]

   \[
   -3x^2 + 6x - 8 = 0
   \]

   \[
   x^2 - 2x + 8 = 0
   \]

   \[
   (x - 2)(x - 4) = 0
   \]

   \( x = 2 \) or \( 4 \)

   \( y = -23 \) or \(-19 \)

   \[
   f''(x) = -6x + 6
   \]

   \[
   f''(2) = +ve, \text{ so a local min. at } (2, -23)
   \]

   \[
   f''(4) = -ve, \text{ so a local max. at } (4, -19)
   \]

   b) (3 marks) Given that \( f(-0.2) = 0 \), sketch the graph of \( f \). Make sure you clearly identify the \( x- \) and \( y- \)intercepts, and the value of the function at the critical points.
14. Scientists at the university have been studying two new strains of influenza, Influenza Pigg and Influenza Birdd. The following sequences represent the number of bacteria each strain produced each week over a 10-week period.

Pigg: \( p_1 = 100, \ p_n = p_{n-1} + 20 \), for \( n = 2, \ldots, 10 \).

Birdd: \( b_1 = 50, \ b_n = \frac{3b_{n-1}}{2} \), for \( n = 2, \ldots, 10 \).

a) (2 marks) Find closed form representations for each of the two sequences.

\[
\begin{align*}
p_n & = 100 + (n-1)20 \\
b_n & = 50 + 20n \\
b_n & = 50 \cdot 1.5^{n-1}
\end{align*}
\]

b) (2 marks) Calculate the value of each of the terms \( p_{10} \) and \( b_{10} \).

\[
\begin{align*}
p_{10} & = 200 + 20 \times 10 \\
b_{10} & = 50 \cdot 1.5^9 \approx 1922
\end{align*}
\]

c) (4 marks) Using the formula for either an arithmetic series or a geometric series, calculate the total number of bacteria produced by Influenza Pigg over the 10 weeks, and calculate the total number of bacteria produced by Influenza Birdd over the 10 weeks.

\[
\begin{align*}
P_n & = \frac{n}{2} \left( 2a + (n-1)d \right) \\
P_{10} & = \frac{10}{2} \left( 2 \times 100 + (10-1)20 \right) \\
& = 1900
\end{align*}
\]

\[
\begin{align*}
B_n & = \frac{a(r^n - 1)}{r-1} \\
B_{10} & = \frac{50 \cdot (\frac{3}{2})^{10} - 1}{\frac{3}{2} - 1} \\
& \approx 5666.5 \\
& \approx 5667
\end{align*}
\]

Questions 15 & 16 see next page.
15. Let \( f(x) = x + 1 \), \( g(x) = x - 1 \), \( h(x) = 2x \), \( n(x) = x^2 \) and \( p(x) = 2x + 1 \).

a) (2 marks) Use composition of functions and one or more of \( f, g, h, n, \) and \( p \) to define a new function \( y(x) = x^2 - 2x + 1 \).

\[
\left(x-1\right)^2 = x^2 - 2x + 1 = n(g(x)) = y(x)
\]

b) (2 marks) Write down the domain and range for the function \( y \).

\[
\begin{align*}
g(x) & : \mathbb{R} \\
R & : \mathbb{R} \\
\text{No restriction on range of } g(x),
\end{align*}
\]

\[
\begin{align*}
n(x) & : \mathbb{R} \\
\text{Domain of } y(x) & \text{ is } \mathbb{R} \\
\text{Range is } [0, \infty)
\end{align*}
\]

16. (3 marks) Determine the inverse function of \( f(x) = \sqrt{2x-1} + 2 \). Simplify your answer.

\[
y = \sqrt{2x-1} + 2
\]

\[
y - 2 = \sqrt{2x-1}
\]

\[
(y-2)^2 = 2x-1
\]

\[
y^2 - 4y + 4 + 1 = 2x
\]

\[
\Rightarrow x = \frac{y^2 - 4y + 5}{2}
\]

\[
\therefore \ f^{-1}(x) = \frac{x^2 - 4x + 5}{2}
\]
17. Find the following integrals

a) (3 marks) \(\int 2\sin(\sqrt{2x} + 3)\,dx\)

\[= -\sqrt{2}\cos\left(\sqrt{2}x + 3\right) + C\]

b) (4 marks) \(\int_2^3 \frac{3x}{(x^2 - 3)^4}\,dx\)

Let \(u = x^2 - 3\)

\[du = 2x\,dx\]

\[dx = \frac{du}{2x}\]

\[\int_2^3 \frac{3x}{u^4} \cdot \frac{du}{2x} = \int_{x=2}^{x=3} \frac{3}{2u^4} \cdot du\]

\[= \left[-\frac{1}{2u^3}\right]_{x=2}^{x=3}\]

\[= \left[-\frac{1}{2(x^2 - 3)^3}\right]_{2}^{3}\]

\[= -\frac{1}{2 \cdot 6^3} - \frac{1}{2 \cdot 1^3} \approx 0.5\]
18. An open rectangular box has a volume of 1000cm³. Its base has dimensions x cm and 2x cm.

a) (1 mark) Show that the height, h, of the box is \( \frac{500}{x^2} \).

\[ V_{\text{box}} = l \times w \times h \]
\[ 1000 = x \times 2x \times h \quad \Rightarrow \quad h = \frac{1000}{2x^2} = \frac{500}{x^2} \]

b) (2 marks) Find an expression for the surface area of the box.

\[ S_A(x) = 2x \times x + 2(xh) + 2(2x \times h) \]
\[ = 2x^2 + 4xh + 4xh \]
\[ = 2x^2 + 6xh \]

c) (4 marks) Find the dimensions of the box that will give the minimum surface area.

\[ S_A(x) = 2x^2 + 6xh \]
\[ = 2x^2 + 6x \times \frac{500}{x^2} \]
\[ = 2x^2 + \frac{3000}{x} \]

\[ S_A'(x) = 4x - \frac{3000}{x^2} \]

\[ S_A''(x) = 4 \times \frac{6000}{x^4} \]

Critical points when \( S_A'(x) = 0 \)
\[ 0 = 4x - \frac{3000}{x^2} \]
\[ 3000 \div x^2 = 4x \]
\[ 750 = x^3 \]
\[ x \approx 9.1 \text{ cm} \]

Question 19 see next page.

- Dimensions are approx.
  9.1 cm, 10.2 cm × 6.0 cm.
19. Let \( w = 4 \text{cis} \frac{3\pi}{6} \) and \( z = (2 - 2i)^{-1} \) be two complex numbers.

a) (3 marks) Find the real and imaginary parts of \( z \).

\[
Z = \frac{1}{2 - 2i} \cdot \frac{2 + 2i}{2 + 2i} = \frac{2 + 2i}{8} = \frac{1}{4} + \frac{1}{4}i,
\]

\[
\text{Re}(z) = \frac{2}{8} = \frac{1}{4}, \quad \text{Im}(z) = \frac{2}{8} = \frac{1}{4}.
\]

b) (3 marks) Write \( z \) in polar form.

\[
Z = \sqrt{\frac{1}{8} \cdot \frac{1}{8}} \cdot \text{cis} \left( \frac{\pi}{4} \right) = \frac{1}{2\sqrt{2}} \cdot \text{cis} \left( \frac{\pi}{4} \right)
\]

\[1 \leq |z| = \sqrt{\left( \frac{1}{8} \right)^2 + \left( \frac{1}{8} \right)^2} = \sqrt{\frac{1}{8}} = \frac{1}{\sqrt{2}} \]

\[
\therefore Z = \frac{1}{2\sqrt{2}} \cdot \text{cis} \left( \frac{\pi}{4} \right)
\]

c) (2 marks) Calculate \( w^3 \).

\[
w^3 = 4^3 \cdot \text{cis} \left( 3 \times \frac{5\pi}{6} \right) = 64 \cdot \text{cis} \left( \frac{15\pi}{6} \right) = 64 \cdot \text{cis} \left( \frac{\pi}{2} \right)
\]

d) (2 marks) Calculate \( \frac{z}{w} \).

\[
\frac{z}{w} = \frac{\frac{1}{2\sqrt{2}} \cdot \text{cis} \left( \frac{\pi}{4} \right)}{4 \cdot \text{cis} \left( \frac{5\pi}{6} \right)} = \frac{1}{8\sqrt{2}} \cdot \text{cis} \left( \frac{\pi}{4} - \frac{5\pi}{6} \right) = \frac{1}{8\sqrt{2}} \cdot \text{cis} \left( -\frac{3\pi}{4} \right)
\]

Question 20 see next page.
20. (6 marks) Use Mathematical Induction to show that 8 divides $3^{2n} - 1$, for integers $n \geq 1$.

Let $P(n)$ be the claim that $3^{2n} - 1$ is divisible by 8.

$n = 1$ 
$3^{2 \cdot 1} - 1 = 3^2 - 1 = 8$

So $P(1)$ holds as $8 \div 8 = 1$

Assume $P(k)$ holds, i.e., $3^{2k} - 1 = 8A$, where $A \in \mathbb{Z}^+$. 

We need to show that $P(k+1)$ also holds.

$P(k+1) = 3^{2(k+1)} - 1$

$= 3^{2k+2} - 1$

$= 3^{2k} \cdot 3^2 - 1$

$= 3^{2k} \cdot (3^2 - 9 + 9) - 1$

$= 3^{2k} \cdot (3^{2k} - 9) + 8$

$= 3^{2k} \cdot P(k) + 8$

$= 9 \cdot 8A + 8 = 8(9A + 1)$

$= 8M$, where $M \in \mathbb{Z}^+$. 

So $P(k+1)$ holds, and 8 divides $3^{2n} - 1$ by mathematical induction.
21. (6 marks) A yacht making 7 knots through the water is headed at a bearing of 240°. If there is a current of 3 knots from the east, what are the yacht’s resultant speed and direction?

\[
\text{Yacht: } -7 \cos 30 \mathbf{i} - 7 \sin 30 \mathbf{j} = -\frac{7 \sqrt{3}}{2} \mathbf{i} - \frac{7}{2} \mathbf{j}
\]

\[
\text{Current: } -3 \mathbf{i}
\]

\[
\text{Resultant: } \left( \frac{7 \sqrt{3}}{2} - 3 \right) \mathbf{i} - \frac{7}{2} \mathbf{j}
\]

\[
\frac{-7 \sqrt{3}}{2} - 3
\]

\[
\therefore \phi = \arctan \left( \frac{-\frac{7}{2}}{-\frac{7 \sqrt{3}}{2} - 3} \right)
\]

\[
\approx 21.1°
\]

\[
\text{Let } r = \text{resultant speed}
\]

\[
|r| = \sqrt{\left( \frac{7 \sqrt{3}}{2} - 3 \right)^2 + \left( -\frac{7}{2} \right)^2}
\]

\[
\approx 9.71 \text{ knots}
\]

\[
\text{Bearing is approx. } 248.9° \text{ or } \approx 568.9° \text{ W.}
\]

Question 22 see next page.
22. a) (3 marks) Find the values of \( x \) such that

\[ |3 + 2x| > 7 - 3x. \]

\[ 3 + 2x > 7 - 3x \]
\[ 5x > 4 \]
\[ x > \frac{4}{5} \]

\[ -(3 + 2x) > 7 - 3x \]
\[ -3 - 2x > 7 - 3x \]
\[ x > 10 \]

\[ \{ x > \frac{4}{5} \mid (\frac{4}{5}, \infty) \} \]

b) (5 marks) Find the values of \( x \) such that \( \left| \frac{1}{2x - 7} \right| \leq \frac{3}{4} \).

\[ \frac{1}{|2x - 7|} \leq \frac{3}{4} \]
\[ 4 \leq 3 |2x - 7| \]
\[ 4 \leq 3(2x - 7) \]
\[ 4 \leq 6x - 21 \]
\[ 6x \geq 25 \]
\[ x \geq \frac{25}{6} \]

\[ \left( -\infty, \frac{17}{6} \right] \cup \left[ \frac{25}{6}, \infty \right) \]

Question 23 see next page.
23. a) (2 marks) Find the derivative of \(-y^3 + 3y = 6x^2 - 2\).

\[-3y^2 \frac{dy}{dx} + 3 \frac{dy}{dx} = 12x\]
\[(2y^2 - 3) \frac{dy}{dx} = 12x\]
\[
\frac{dy}{dx} = \frac{12x}{-3y^2 + 3} = \frac{4x}{-y^2 + 1}
\]

b) (3 marks) Hence find the equation of the tangent line to the graph of \(-y^3 + 3y = 6x^2 - 2\) at the point \(A = (0, 2)\).

At \(A(0, 2)\), \[
\frac{dy}{dx} = \frac{4 \times 0}{-2^2 + 1} = 0
\]

Tangent line is of form \(y = mx + c\)

At \(A\), \(m = 0\), so \(y = 0 \times 0 + c\)
\[2 = 0 \times 0 + c\]
\[\therefore c = 2\]

\[
\therefore \text{Tangent line is } y = 2.
\]
24. Let \( A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 2 & 5 & 3 \end{pmatrix} \) and \( B = \begin{pmatrix} -40 & 9 & a \\ 13 & b & -5 \\ c & -1 & -2 \end{pmatrix} \).

a) (3 marks) Determine the values of \( a, b \) and \( c \) such that \( A = B^{-1} \).

\[ A \cdot B^{-1} = I \Rightarrow -40 + 2b + 3c = 1 \]

\[ \therefore c = 5 \]

\[ q \cdot b - 3 = 0 \]

\[ \Rightarrow b = -3 \]

\[ q \cdot 10 - 6 = 0 \]

\[ \Rightarrow q = 1/6. \]

b) (3 marks) Use the result from part (a) to solve the following system of equations.

\[
\begin{align*}
x + 2y + 3z &= 0 \\
x + 8z &= 2 \\
2x + 5y + 3z &= -1
\end{align*}
\]

\[
\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}
\]

\[
\begin{pmatrix} -40 & 9 & 16 \\ 13 & -3 & -5 \\ 5 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -40 & 9 & 16 \\ 13 & -3 & -5 \\ 5 & -1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}
\]

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}
\]

\[ x = -2, \quad y = -1, \quad z = 0. \]