MATH2000 ASSIGNMENT 1 SOLUTIONS.

Q1. Rewrite the equation as

$$\frac{dy}{dx} = -\frac{2x+4y}{4x-2y} = -\left(\frac{x+2y}{2x-y}\right) = f(x,y)$$

Domain of $f(x,y)$ is $D = \mathbb{R}^2 \setminus S$,

where $S = \{(x,y) \mid y = 2x\}$. That is D contains all points in $\mathbb{R}^2$ but not in S. The function $f(x,y)$ is continuous everywhere in D, so there exist solution to the IVP with $y(x_0)=y_0$ insides a rectangle containing (x_0,y_0) and only points in D.
Also $\frac{\partial f}{\partial y} = \frac{-5x}{(y-2x)^2}$, whose domain is also D.
From the result in lectures, any rectangle that contains a solution passing through (x_0,y_0) must only contain one solution.
Let $(x_0,y_0) \in D$.
Set $P(x,y) = 2x+4y$ & $Q(x,y) = 4x-2y$.
The equation can be expressed as

$$P + Q\frac{dy}{dx} = 0.$$

$$\frac{\partial P}{\partial y} = 4, \quad \frac{\partial Q}{\partial x} = 4 \Rightarrow \text{equation is exact.}$$

$$\Rightarrow \exists f(x,y) \text{ s.t. } \frac{\partial f}{\partial x} = P \ \& \ \frac{\partial f}{\partial y} = Q$$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x+4y \Rightarrow f(x,y) = x^2+4xy+g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 4x + g'(y) = Q = 4x-2y.$$

$$\Rightarrow g'(y) = -2y \Rightarrow g(y) = -y^2$$

$$\Longrightarrow f(x,y) = x^2+4xy-y^2 \quad \text{(up to a constant)}$$

$$\Longrightarrow x^2+4xy-y^2 = C$$ gives the implicit solution to the ODE.

For $y(x_0)=y_0$ with $(x_0,y_0)\in D$.

$$\Longrightarrow C = x_0^2+4x_0y_0-y_0^2$$

$$\Rightarrow x^2+4xy-y^2 = x_0^2+4x_0y_0-y_0^2$$ is the implicit solution to the IVP. (You may choose explicit x_0, y_0.

Notes & further discussion.

We can treat this as a quadratic in y:

$$y^2 - 4xy - x^2 + x_0^2 + 4x_0y_0 - y_0^2 = 0$$

quadratic formula

$$\Rightarrow y = \frac{4x \pm \sqrt{16x^2 - 4(-x^2 + x_0^2 + 4x_0y_0 - y_0^2)}}{2}$$

$$= 2x \pm \sqrt{5x^2 - x_0^2 - 4x_0y_0 + y_0^2}$$

or $$y = 2x \pm \sqrt{5x^2 - (x_0^2 + 4x_0y_0 - y_0^2)}$$

~~so~~ if $x^2 < \frac{1}{5}(x_0^2 + 4x_0y_0 - y_0^2)$ there will not be a solution. However, in the case $x_0^2 + 4x_0y_0 - y_0^2 \leq 0$, then there will always be a solution (i.e. for any x). Since $x_0^2 + 4x_0y_0 - y_0^2$ is constant (given x_0 & y_0), for large $|x|$, the solutions approach the asymptotes

$$y = 2x \pm \sqrt{5x^2} \quad \text{or} \quad y = (2+\sqrt{5})x$$
$$\& \quad y = (2-\sqrt{5})x$$

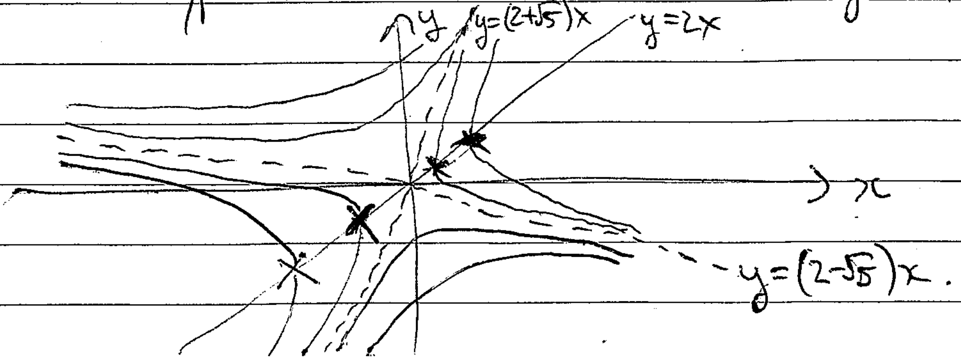
which themselves satisfy the ODE.

(Set $y = kx \Rightarrow y' = k \Rightarrow 2x + 4y + (4x - 2y)\frac{dy}{dx}$

$$= 2x + 4kx + (4x - 2kx)k$$
$$= (2 + 8k - 2k^2)x$$

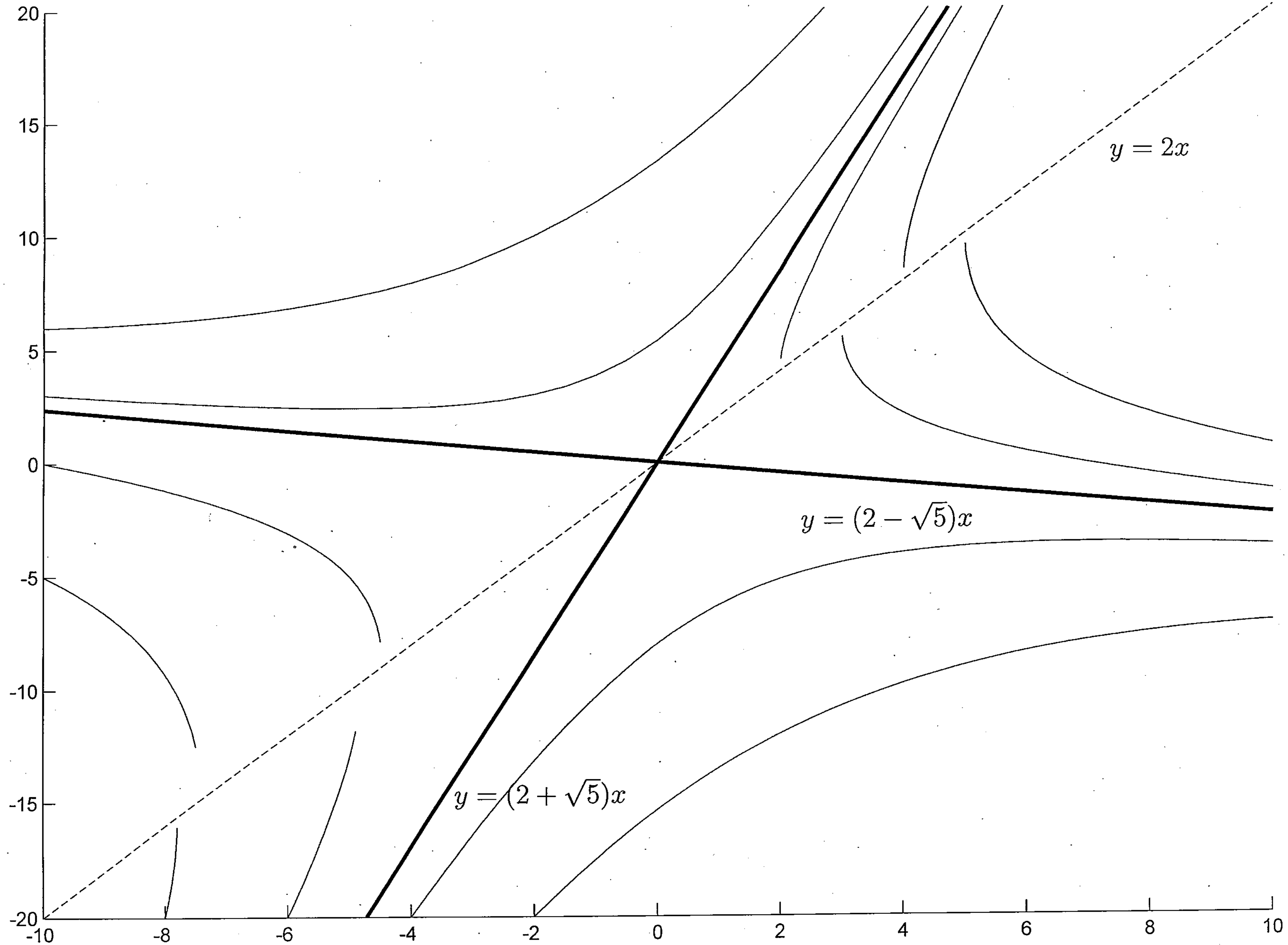
which $= 0$ for $k = 2 \pm \sqrt{5}$.)

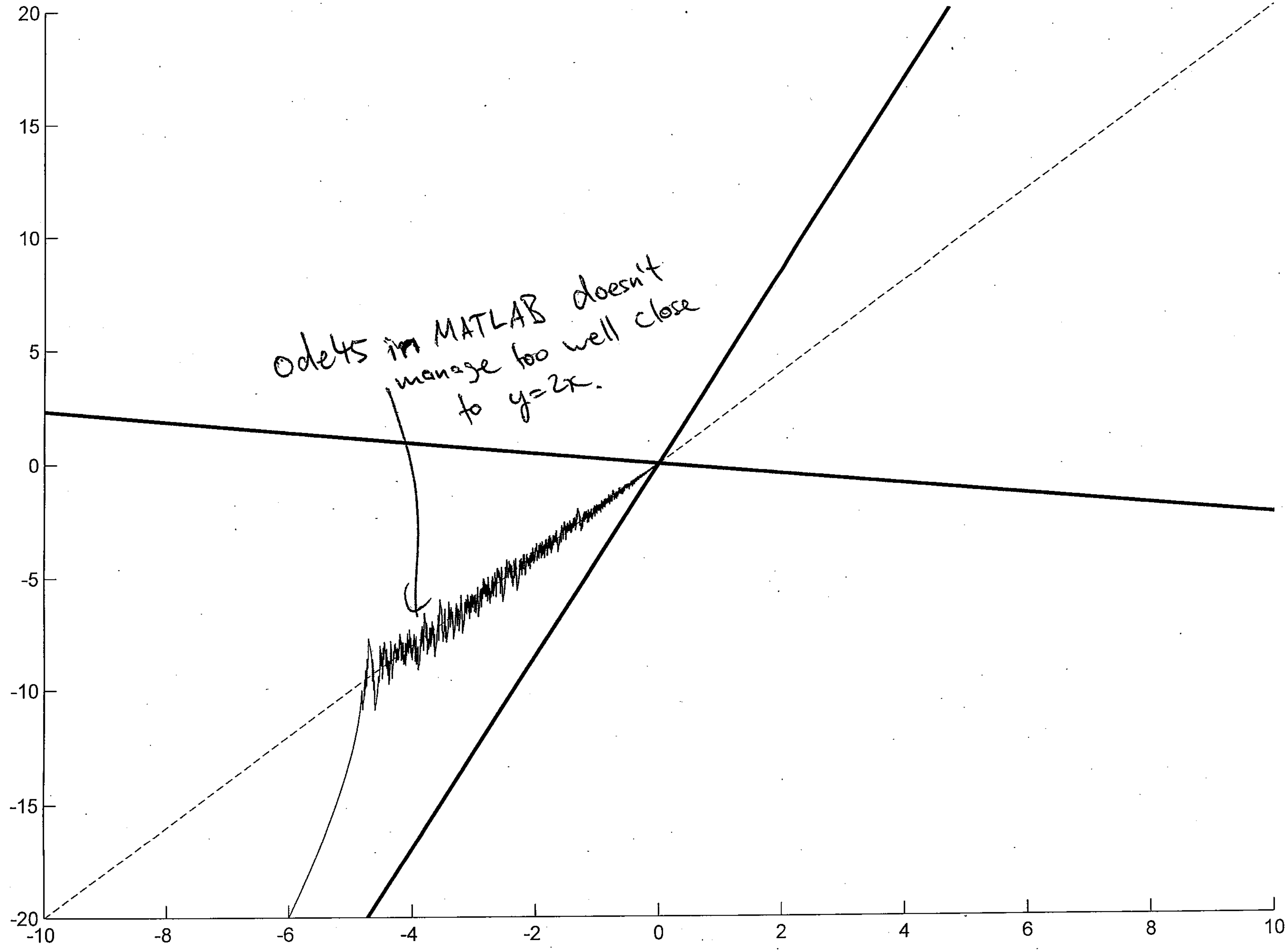
Solutions are hyperbolas which do not cross $y = 2x$.



(see over)

From MATLAB:




ode45 in MATLAB doesn't manage too well close to y=2x.
20
15
10
5
0
-5
-10
-15
-20
-10
-8
-6
-4
-2
0
2
4
6
8
10

Q2. We first note that the general solution is of the form $y = y_H + y_P$, where y_P is a particular solution and y_H is the general solution of the associated homogeneous equation: $y'' + 4y' + 4y = 0$.

The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$

$$\Rightarrow \lambda = -2$$

$$\Rightarrow y_1 = e^{-2x} \quad \& \quad y_2 = xe^{-2x}.$$

$$\Rightarrow y_H = Ae^{-2x} + Bxe^{-2x}.$$

For y_P: The right hand side $= 2e^{-2x} + \dfrac{e^{-2x}}{x^2}$

For $2e^{-2x}$ we guess $y_{P_1} = ax^2e^{-2x}$ (modified method of undetermined coefficients)

$$y_{P_1}' = 2axe^{-2x} - 2ax^2e^{-2x}$$

$$y_{P_1}'' = 2ae^{-2x} - 8axe^{-2x} + 4ax^2e^{-2x}$$

$$\Rightarrow y_{P_1}'' + 4y_{P_1}' + 4y_{P_1} = 2ae^{-2x} - 8axe^{-2x} + 4ax^2e^{-2x} + 8axe^{-2x} - 8ax^2e^{-2x} + 4ax^2e^{-2x} = 2ae^{-2x}$$

$$= \text{r.h.s.}(1) = 2e^{-2x} \Rightarrow a = 1 \Rightarrow y_{P_1} = x^2e^{-2x}.$$

For $r_2 = \dfrac{e^{-2x}}{x^2}$, use variation of parameters, $y_{P_2} = uy_1 + vy_2$

where $u = -\displaystyle\int \frac{y_2 r_2}{W}\,dx$ & $v = \displaystyle\int \frac{y_1 r_2}{W}\,dx$ (from lectures)

with $W = y_1y_2' - y_1'y_2 = e^{-2x}(e^{-2x} - 2xe^{-2x}) - (-2e^{-2x})xe^{-2x}$
$= e^{-4x}$.

$$\Rightarrow u = -\int \frac{xe^{-2x}}{e^{-4x}} \cdot \frac{e^{-2x}}{x^2}\,dx = -\int \frac{dx}{x} = -\ln x.$$

$$v = \int \frac{e^{-2x}}{e^{-4x}} \cdot \frac{e^{-2x}}{x^2}\,dx = \frac{-1}{x}$$

$$\Rightarrow y_{P_2} = -\ln x\, e^{-2x} - \underbrace{\frac{1}{x} \cdot xe^{-2x}}$$

can ignore this since it is proportional to y_1

$\Rightarrow$ general solution is

$$y = Ae^{-2x} + Bxe^{-2x} + x^2e^{-2x} - e^{-2x}\ln x .$$

$$y' = -2Ae^{-2x} + Be^{-2x} - 2Bxe^{-2x} + 2xe^{-2x} - 2x^2e^{-2x} + 2e^{-2x}\ln x - \frac{e^{-2x}}{x}$$

$$y(1) = Ae^{-2} + Be^{-2} + e^{-2} - 0 = 0 \quad \text{---} \ (1)$$

$$y'(1) = -2Ae^{-2} + Be^{-2} - 2Be^{-2} + 2e^{-2} - 2e^{-2} - e^{-2} = 0 \quad \text{----} \ (2)$$

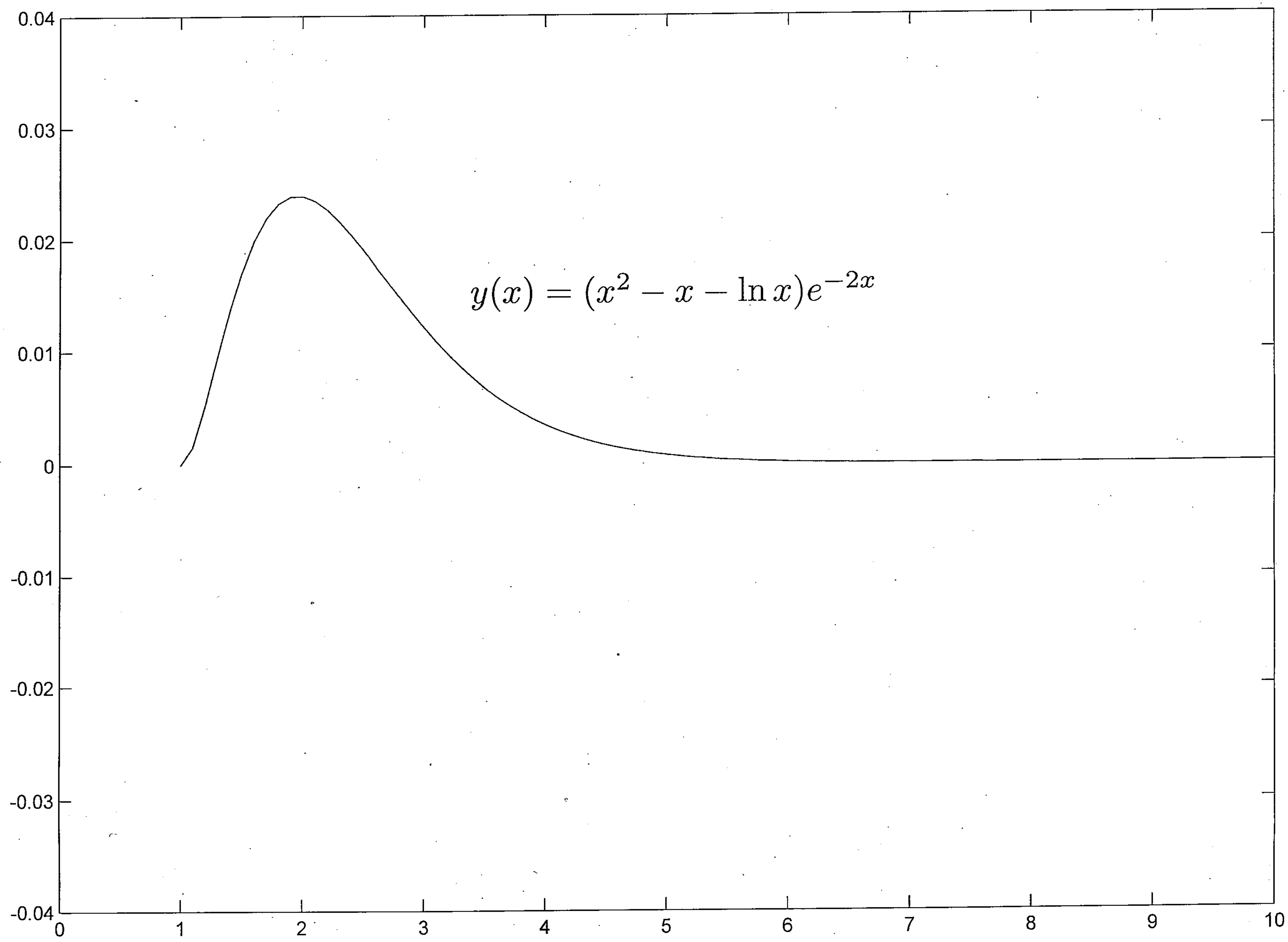
Equ. (1) above implies $\quad A + B + 1 = 0$

" (2) " " $\quad -2A - B - 1 = 0 \quad \Rightarrow B = -2A - 1$

(sub. into (1)) $\Rightarrow \quad A - 2A - 1 + 1 = 0$

$\Rightarrow A = 0$ & $B = -1$.

$\Rightarrow \quad y(x) = (x^2 - x - \ln x)e^{-2x}$. solves the IVP.



$y(x) = (x^2 - x - \ln x)e^{-2x}$
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
0
1
2
3
4
5
6
7
8
9
10

Q3. $y = \text{sech}^{-1} x$. $x \in (0, 1]$ & $y \in [0, \infty)$

$$\Rightarrow x = \text{sech}\, y = \frac{1}{\cosh y} = \frac{2}{e^{y} + e^{-y}}$$

$$\Rightarrow x\left(e^{y} + e^{-y}\right) = 2$$

(mult. e^{y}) $\Rightarrow x(e^{y})^2 - 2e^{y} + x = 0$.

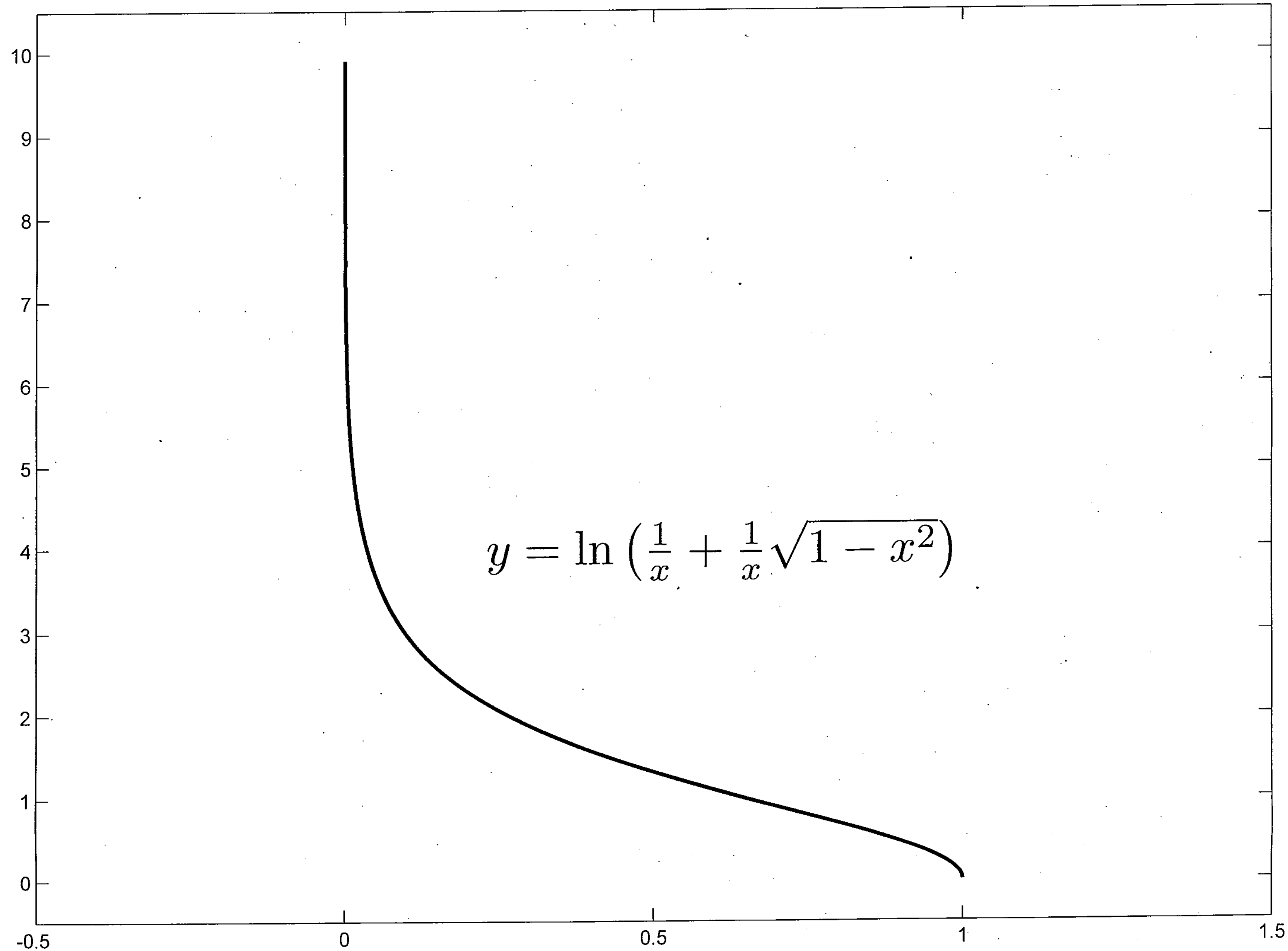
quadratic in e^{y} $\Rightarrow e^{y} = \dfrac{2 \pm \sqrt{4 - 4x^2}}{2x}$

$$\Rightarrow e^{y} = \frac{1}{x} \pm \frac{1}{x}\sqrt{1 - x^2}.$$

Since $y \geq 0 \Rightarrow e^{y} \geq 1$, can only have "+" so

$$e^{y} = \frac{1}{x} + \frac{1}{x}\sqrt{1 - x^2}$$

$$\Rightarrow y = \ln\left(\frac{1}{x} + \frac{1}{x}\sqrt{1 - x^2}\right)$$



$y = \ln\left(\frac{1}{x} + \frac{1}{x}\sqrt{1-x^2}\right)$
0
1
2
3
4
5
6
7
8
9
10
-0.5
0
0.5
1
1.5

Q4. $\displaystyle\int_{-2/3}^{-1/3} \frac{dx}{x\sqrt{1-x^2}}$

Set $x = -\operatorname{sech} t$

$\Rightarrow \dfrac{dx}{dt} = -\dfrac{d}{dt}\left(\dfrac{1}{\cosh t}\right)$

$= \dfrac{1}{\cosh^2 t} \sinh t$

$= \operatorname{sech} t \,.\, \tanh t$

Also, since $\dfrac{\cosh^2 t}{\cosh^2 t} - \dfrac{\sinh^2 t}{\cosh^2 t} = \dfrac{1}{\cosh^2 t}$

$\Rightarrow 1 - \tanh^2 t = \operatorname{sech}^2 t$

$\Rightarrow 1 - \operatorname{sech}^2 t = \tanh^2 t$

$\Rightarrow \sqrt{1-x^2} = \sqrt{1-\operatorname{sech}^2 t} = \sqrt{\tanh^2 t} = \tanh t$ for $t > 0$

Also, $x = -2/3 \Rightarrow t = \operatorname{sech}^{-1}\left(\frac{2}{3}\right)$

$\simeq 0.9624$

& $x = -1/3 \Rightarrow t = \operatorname{sech}^{-1}\left(\frac{1}{3}\right)$

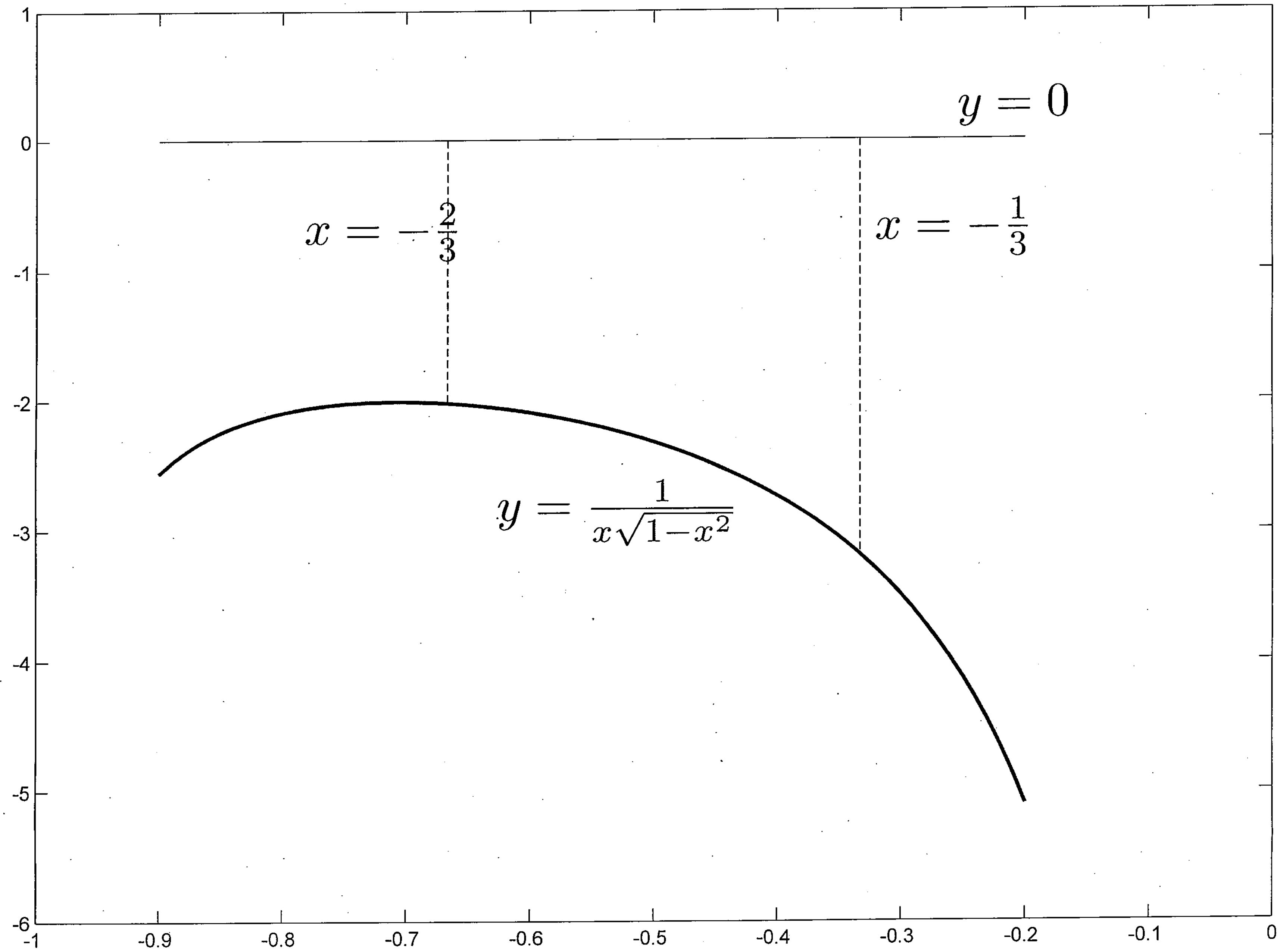
$\simeq 1.7627$

$\Rightarrow \displaystyle\int_{-2/3}^{-1/3} \frac{dx}{x\sqrt{1-x^2}} = \int_{0.9624}^{1.7627} \frac{\operatorname{sech} t \,.\, \tanh t}{-\operatorname{sech} t \,.\, \tanh t}\, dt.$

$= -(1.7627 - 0.9624)$

$= -0.8003$

Note that we can even evaluate sech^{-1} on a calculator using the answer to Q3.



1
0
-1
-2
-3
-4
-5
-6
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
$y = 0$
$x = -\frac{2}{3}$
$x = -\frac{1}{3}$
$y = \frac{1}{x\sqrt{1-x^2}}$