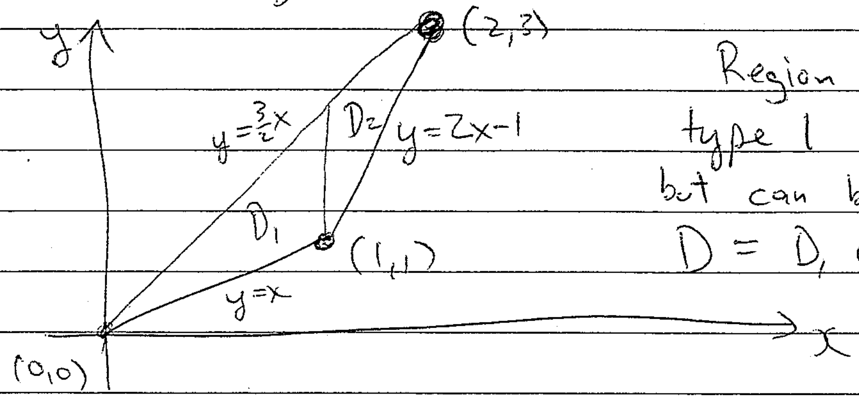
ASSIGNMENT 2 SOLUTIONS.

Q1. Volume of solid above the region D & below $z = 6x + 24y$ is

$$\text{vol.} = \iint_D (6x + 24y)\, dA$$



Region D is neither type 1 nor type 2, but can be expressed as $D = D_1 \cup D_2$, where...

$$D_1 = \{(x,y) \mid 0 \le x \le 1,\ x \le y \le \tfrac{3}{2}x\}$$

$$\& \quad D_2 = \{(x,y) \mid 1 \le x \le 2,\ 2x-1 \le y \le \tfrac{3}{2}x\}.$$

$$\text{so vol.} = \iint_{D_1} (6x+24y)\, dA + \iint_{D_2} (6x+24y)\, dA$$

$$= \int_0^1 \int_x^{\frac{3}{2}x} (6x+24y)\, dy\, dx + \int_1^2 \int_{2x-1}^{\frac{3}{2}x} (6x+24y)\, dy\, dx$$

$$= \int_0^1 \Big[6xy + 12y^2 \Big]_{y=x}^{y=\frac{3}{2}x} dx + \int_1^2 \Big[6xy + 12y^2 \Big]_{y=2x-1}^{y=\frac{3}{2}x} dx$$

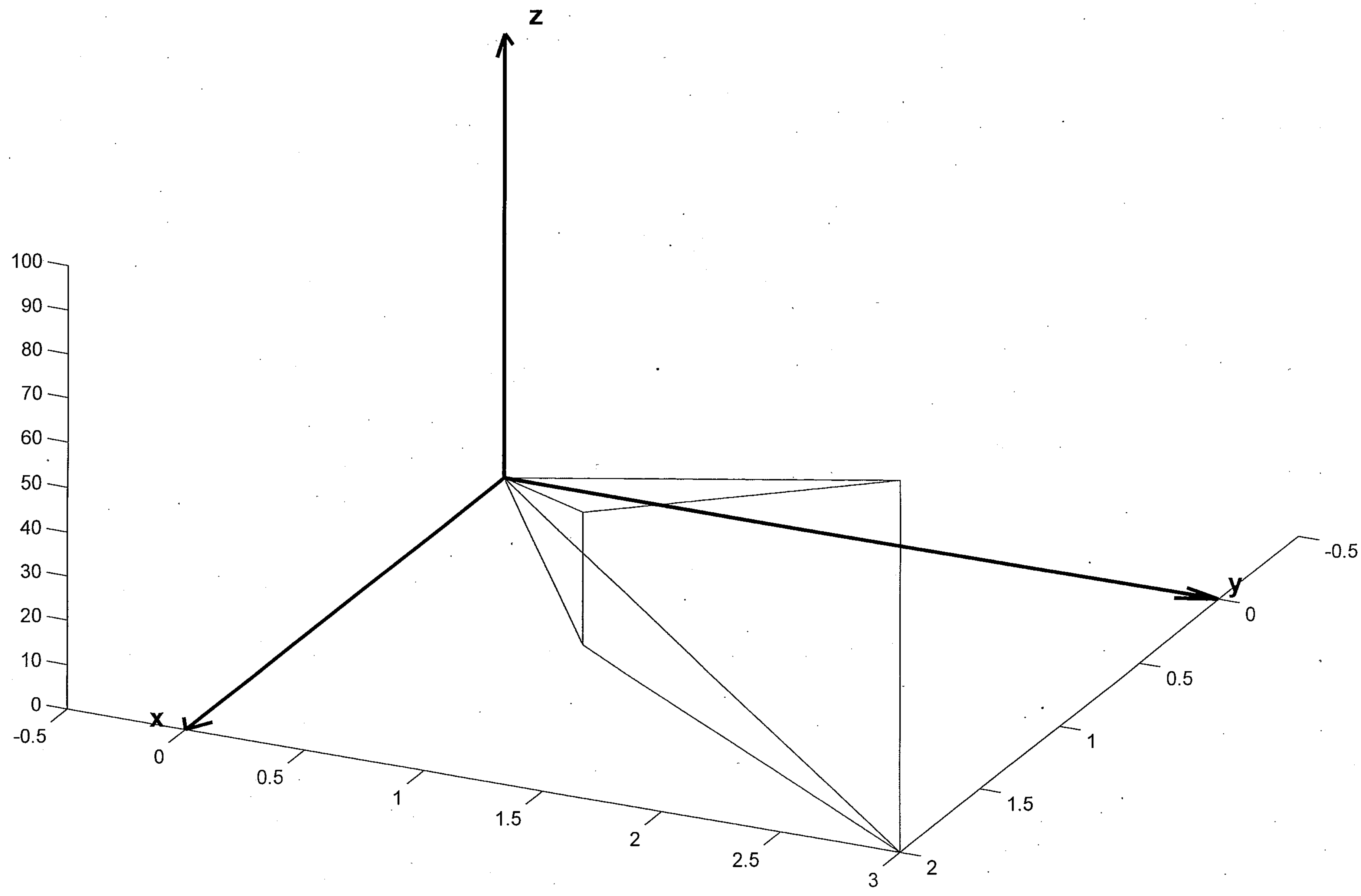
$$= \int_0^1 (9x^2 + 27x^2 - 6x^2 - 12x^2)\, dx + \int_1^2 \big(9x^2 + 27x^2 - 6x(2x-1) - 12(4x^2-4x+1)\big)\, dx$$

$$= \int_0^1 18x^2\, dx + \int_1^2 (-24x^2 + 54x - 12)\, dx$$

$$= \Big[6x^3\Big]_0^1 + \Big[-8x^3 + 27x^2 - 12x\Big]_1^2$$

$$= 6 + -64 + 108 - 24 - (-8 + 27 - 12)$$

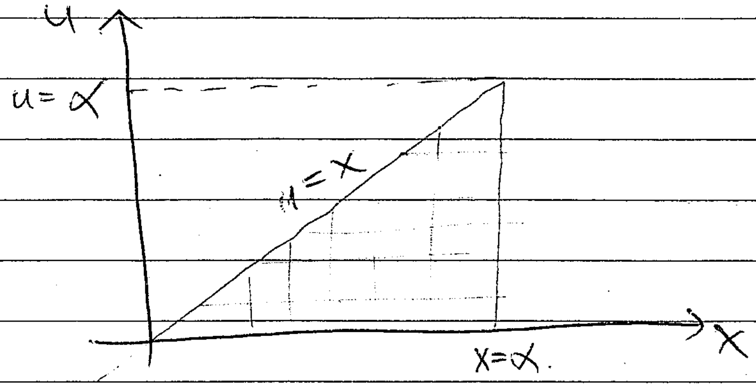
$$= 19$$


z
100
90
80
70
60
50
40
30
20
10
0
-0.5
x
0
0.5
1
1.5
2
2.5
3
y
-0.5
0
0.5
1
1.5
2

Q2. $\displaystyle\int_0^{\alpha} \operatorname{erf}(x)\,dx = \frac{2}{\sqrt{\pi}}\int_0^{\alpha}\int_0^{x} e^{-u^2}\,du\,dx$

Region of integration $R = \{(x,u) \mid 0 \le x \le \alpha,\ 0 \le u \le x\}$.

Swap order of integration:



$\Rightarrow R = \{(x,u) \mid 0 \le u \le \alpha,\ u \le x \le \alpha\}$.

$$\Longrightarrow \int_0^{\alpha} \operatorname{erf}(x)\,dx = \frac{2}{\sqrt{\pi}}\int_0^{\alpha}\int_u^{\alpha} e^{-u^2}\,dx\,du$$

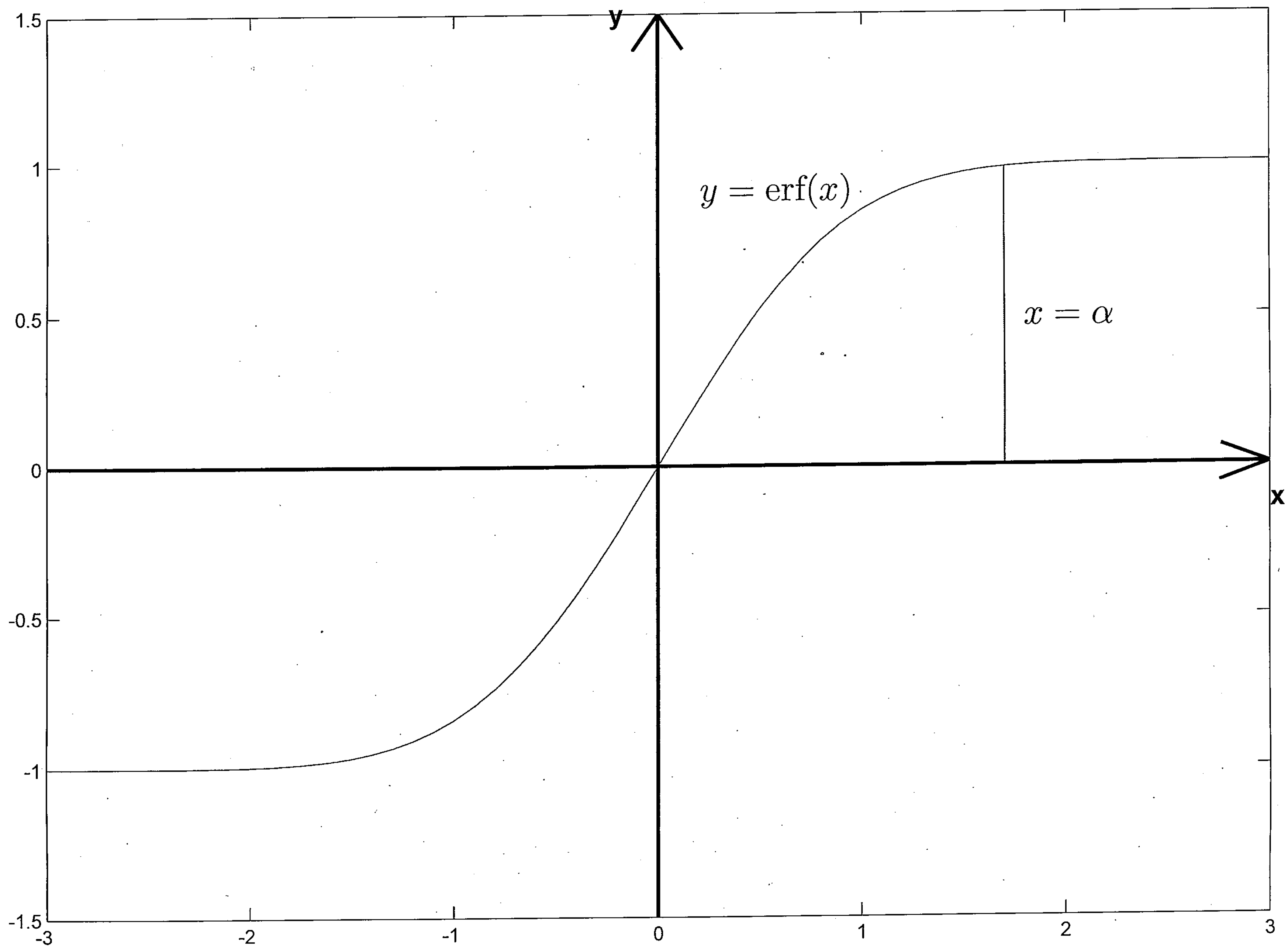
$$= \frac{2}{\sqrt{\pi}}\int_0^{\alpha}\Big[x e^{-u^2}\Big]_u^{\alpha}\,du$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\alpha}\left(\alpha e^{-u^2} - u e^{-u^2}\right)du$$

$$= \frac{2\alpha}{\sqrt{\pi}}\int_0^{\alpha} e^{-u^2}\,du + \frac{1}{\sqrt{\pi}}\int_0^{\alpha}(-2u)e^{-u^2}\,du$$

$$= \alpha\operatorname{erf}(\alpha) + \frac{1}{\sqrt{\pi}}\int_{t=0}^{t=-\alpha^2} e^{t}\,dt \qquad \begin{pmatrix} t = -u^2, \\ dt = -2u\,du \end{pmatrix}$$

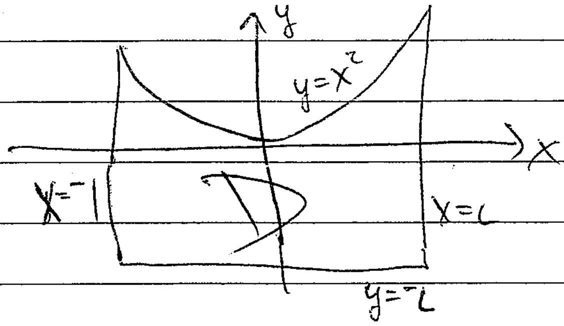
$$= \alpha\operatorname{erf}(\alpha) + \frac{1}{\sqrt{\pi}}\left(e^{-\alpha^2} - 1\right)$$


y
x
$y = \mathrm{erf}(x)$
$x = \alpha$
1.5
1
0.5
0
-0.5
-1
-1.5
-3
-2
-1
0
1
2
3

Q3. Centre of mass has coordinates $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\iint_D \rho x \, dA}{\iint_D \rho \, dA}, \quad \bar{y} = \frac{\iint_D \rho y \, dA}{\iint_D \rho \, dA}.$$

In this case ρ is a constant & from the diagram



The lamina occupies the region

$$D = \{(x,y) \mid -1 \le x \le L, \ -L \le y \le x^2\}$$

So in this case

$$\bar{x} = \frac{\iint_D x \, dA}{\iint_D dA}, \quad \bar{y} = \frac{\iint_D y \, dA}{\iint_D dA} \qquad \text{(constant } \rho \text{ cancels)}$$

$$\iint_D dA = \int_{-1}^{L} \int_{-L}^{x^2} dy \, dx$$

$$= \int_{-1}^{L} \Big[y \Big]_{y=-L}^{y=x^2} dx$$

$$= \int_{-1}^{L} (x^2 + L) \, dx$$

$$= \left[\tfrac{1}{3} x^3 + xL \right]_{-1}^{L}$$

$$= \tfrac{1}{3} L^3 + L^2 + \tfrac{1}{3} + L$$

$$= \tfrac{1}{3} (L+1)^3$$

$$\iint_D x\,dA = \int_{-1}^{L}\int_{-L}^{x^2} x\,dy\,dx$$

$$= \int_{-1}^{L}\Big[x\,y\Big]_{y=-L}^{y=x^2}\,dx$$

$$= \int_{-1}^{L}\left(x^3 + Lx\right)dx$$

$$= \frac{1}{4}x^4 + \frac{1}{2}Lx^2\Big|_{-1}^{L}$$

$$= \frac{1}{4}L^4 + \frac{1}{2}L^3 - \frac{1}{4} - \frac{1}{2}L$$

$$= \frac{1}{4}\left(L^4 + 2L^3 - 2L - 1\right)$$

$$= \frac{1}{4}(L-1)(L+1)^3$$

$$\iint_D y\,dA = \int_{-1}^{L}\int_{-L}^{x^2} y\,dy\,dx$$

$$= \int_{-1}^{L}\left[\frac{1}{2}y^2\right]_{y=-L}^{y=x^2}dx$$

$$= \int_{-1}^{L}\left(\frac{1}{2}x^4 - \frac{1}{2}L^2\right)dx$$

$$= \frac{1}{10}x^5 - \frac{1}{2}L^2x\Big|_{-1}^{L}$$

$$= \frac{1}{10}L^5 - \frac{1}{2}L^3 + \frac{1}{10} - \frac{1}{2}L^2$$

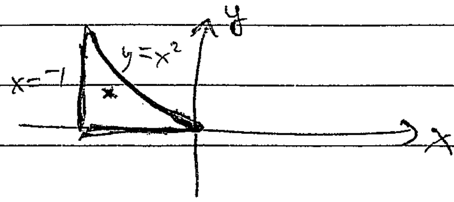
$$= \frac{1}{10}\left(L^5 - 5L^3 - 5L^2 + 1\right)$$

$$= \frac{1}{10}\left(L^2 - 3L + 1\right)(L+1)^3.$$

$$\Rightarrow \bar{x} = \frac{\frac{1}{4}(L-1)(L+1)^3}{\frac{1}{3}(L+1)^3} = \frac{3}{4}(L-1)$$

$$\bar{y} = \frac{\frac{1}{10}(L^2-3L+1)(L+1)^3}{\frac{1}{3}(L+1)^3} = \frac{3}{10}(L^2-3L+1)$$

At $L=0$, $\bar{x} = \frac{-3}{4}$, $\bar{y} = \frac{3}{10}$ which lies inside the region:



As L increases from 0, we note $\bar{x}$ will increase, therefore never cross $x=-1$.

Does it ever cross $y=x^2$?

$$\bar{x}^2 = \frac{9}{16}(L-1)^2 = \frac{9}{16}(L^2-2L+1)$$

$$\text{set} = \bar{y} = \frac{3}{10}(L^2-3L+1)$$

$$\Rightarrow \left(\frac{9}{16}-\frac{3}{10}\right)L^2 + \left(\frac{9}{10}-\frac{9}{8}\right)L + \frac{9}{16}-\frac{3}{10} = 0$$

$$\Rightarrow \left(\frac{90-48}{160}\right)L^2 + \left(\frac{72-90}{80}\right)L + \frac{90-48}{160} = 0.$$

$$\Rightarrow 42L^2 - 36L + 42 = 0$$

or $7L^2 - 6L + 7 = 0$

$$\Rightarrow L = \frac{6 \pm \sqrt{36-4\times 7\times 7}}{14} \Rightarrow \text{no real solutions}$$

So as L increases, centre of mass never crosses $y=x^2$.

Does it ever cross $y=-L$?

i.e. set $\bar{y} = -L \Rightarrow \frac{3}{10}(L^2-3L+1) = -L$

$$\Rightarrow \frac{3}{10}L^2 + \frac{1}{10}L + \frac{3}{10} = 0$$

$$\Rightarrow 3L^2 + L + 3 = 0$$

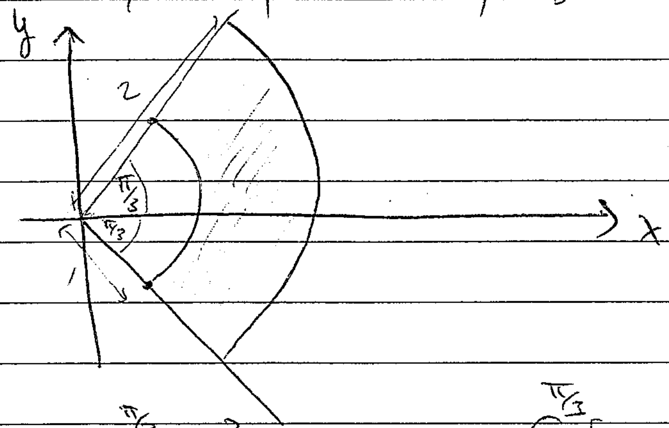
$$\Rightarrow L = \frac{-1 \pm \sqrt{1-4\times 3\times 3}}{6} \Rightarrow \text{no real solutions.}$$

Also, centre of mass never crosses $x=L$ since $\bar{x} = \frac{3}{4}(L-1) = L$ has no solution for $L \geq 0$.

Therefore the centre of mass always remains inside the region.

Q4. From $\int_{-\pi/3}^{\pi/3}\int_{1}^{2} r\,dr\,d\theta$, the region of integration is

$$\{(r,\theta) \mid 1 \le r \le 2,\ -\pi/3 \le \theta \le \pi/3\} \quad \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$$



Also,

$$\int_{-\pi/3}^{\pi/3}\int_{1}^{2} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=1}^{r=2} d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(4-1)\,d\theta$$

$$= \frac{3}{2}\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = \pi.$$