Q1. (a) $I = \int_0^1 \int_0^{2z} \left( \int_z^1 dx \right) dy\, dz + \int_0^1 \int_{2z}^{1+z} \left( \int_{y-z}^1 dx \right) dy\, dz$

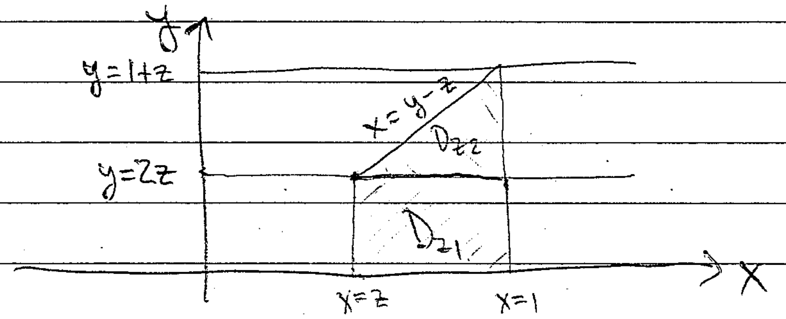
$= \int_0^1 \int_0^{2z} (1-z)\, dy\, dz \; + \int_0^1 \int_{2z}^{1+z} (1 - y + z)\, dy\, dz$

$= \int_0^1 \Big[ (1-z) y \Big]_{y=0}^{y=2z} dz + \int_0^1 \Big[ y - \tfrac{1}{2} y^2 + yz \Big]_{y=2z}^{y=1+z} dz$

$= \int_0^1 \left( 2z - 2z^2 \right) dz + \int_0^1 \left( 1 + \cancel{z} - \tfrac{1}{2}(1 + 2z + z^2) + \cancel{z} + z^2 - \cancel{2z} + 2z^2 - 2z^2 \right) dz$

$= \int_0^1 \left( \tfrac{1}{2} + z - \tfrac{3}{2} z^2 \right) dz = \tfrac{1}{2} z + \tfrac{1}{2} z^2 - \tfrac{1}{2} z^3 \Big|_0^1 = \tfrac{1}{2}.$

(b) In x-y plane:




$I = \iiint_{V_1} dV + \iiint_{V_2} dV$

$= \int_0^1 \left( \iint_{D_{z1}} dA \right) dz + \int_0^1 \left( \iint_{D_{z2}} dA \right) dz$

For $z \in [0,1]$ fixed,

$D_{z1} = \{ (x,y) \mid 0 \le y \le 2z, \; z \le x \le 1 \}$.

$D_{z2} = \{ (x,y) \mid 2z \le y \le 1+z, \; y - z \le x \le 1 \}$.

From diagram,

$D_{z1} \cup D_{z2} = \{ (x,y) \mid z \le x \le 1, \; 0 \le y \le x + z \}$

$$\Rightarrow I = \int_0^1 \int_z^1 \int_0^{x+z} dy\, dx\, dz$$

$$= \int_0^1 \int_z^1 (x+z)\, dx\, dz$$

$$= \int_0^1 \left[ \tfrac{1}{2}x^2 + xz \right]_{x=z}^{x=1} dz$$

$$= \int_0^1 \left( \tfrac{1}{2} + z - \tfrac{1}{2}z^2 - z^2 \right) dz.$$

$$= \int_0^1 \left( \tfrac{1}{2} + z - \tfrac{3}{2}z^2 \right) dz = \tfrac{1}{2} \text{ as before.}$$

Q2. The region can be expressed as

$$T = \{(r,\theta,\phi) \mid 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi,\ 0 \le r \le 2\sin\phi\}$$

$$\text{vol} = \iiint_T dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2\sin\phi} r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{3}r^3\right]_{r=0}^{r=2\sin\phi} \sin\phi \, d\phi\, d\theta$$

$$= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi} \sin^4\phi \, d\phi\, d\theta.$$

Use trig. ident. $\sin^2\phi = \frac{1}{2} - \frac{1}{2}\cos 2\phi$

$$\Rightarrow \sin^4\phi = \left(\frac{1}{2} - \frac{1}{2}\cos 2\phi\right)\left(\frac{1}{2} - \frac{1}{2}\cos 2\phi\right)$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2\phi + \frac{1}{4}\cos^2 2\phi$$

Also note trig. ident. $\cos^2 2\phi = \frac{1}{2} + \frac{1}{2}\cos 4\phi$.

$$\Rightarrow \sin^4\phi = \frac{1}{4} - \frac{1}{2}\cos 2\phi + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4\phi\right)$$

$$= \frac{3}{8} - \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi.$$

$$\Rightarrow \text{vol.} = \frac{8}{3}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\left(\frac{3}{8} - \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi\right) d\phi.\right)$$

$$= \frac{8}{3} \times 2\pi \times \left[\frac{3}{8}\phi - \underbrace{\frac{1}{4}\sin 2\phi}_{=0} + \underbrace{\frac{1}{32}\sin 4\phi}_{=0}\right]_{\phi=0}^{\phi=\pi}$$

$$= 2\pi^2.$$

Q3. $\text{vol} = \iiint_T dV$, $T = \{(u,v,w) \mid 0 \le u \le 1,\ 0 \le w \le 2\pi,\ 0 \le v \le 2\pi\}$.

$$= \iiint_T (\text{Jacobian})\, du\, dv\, dw.$$

where $\text{Jacobian} = \left| \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \right|$

$$= \left| \det \begin{pmatrix} \sin v \cos w & u \cos v \cos w & -(h + u\sin v)\sin w \\ \sin v \sin w & u \cos v \sin w & (h + u \sin v)\cos w \\ \cos v & -u \sin v & 0 \end{pmatrix} \right|$$

(expand along third row)

$$= \left| \cos v \begin{vmatrix} u\cos v \cos w & -(h+u\sin v)\sin w \\ u \cos v \sin w & (h+u\sin v)\cos w \end{vmatrix} + u\sin v \begin{vmatrix} \sin v \cos w & -(h+u\sin v)\sin w \\ \sin v \sin w & (h+u\sin v)\cos w \end{vmatrix} \right|$$

$$= \cos v \left( u\cos v \cos^2 w (h+u\sin v) + u \cos v \sin^2 w (h + u\sin v) \right) + u\sin v \left( \sin v \cos^2 w (h+u\sin v) + \sin v \sin^2 w (h+u\sin v) \right)$$

$$= u\cos^2 v (h+u\sin v) + u\sin^2 v (h+u\sin v)$$

$$= u(h+u\sin v)$$

$$\Rightarrow \text{vol.} = \int_0^{2\pi}\int_0^{2\pi}\int_0^1 u(h+u\sin v)\, du\, dv\, dw$$

$$= \int_0^{2\pi}\int_0^{2\pi} \left[ \tfrac{1}{2}hu^2 + \tfrac{1}{3}u^3 \sin v \right]_{u=0}^{u=1} dv\, dw$$

$$= \int_0^{2\pi}\int_0^{2\pi} \left( \tfrac{1}{2}h + \tfrac{1}{3}\sin v \right) dv\, dw$$

$$= \int_0^{2\pi} \left[ \tfrac{1}{2}hv - \tfrac{1}{3}\cos v \right]_{v=0}^{v=2\pi} dw$$

$$= \int_0^{2\pi} \pi h\, dw = 2\pi^2 h.$$

(Compare answer with Q2 for $h=1$)

<u>Note:</u> $h=0$ gives spherical coord. transformation & recovers Jacobian of spherical coords. Why not vol.? For $0 \le h < 1$ bounds not as simple as those given

Q4 Calculate the moment of inertia of each object about the axis of rotation.

Solid Cylinder : Take the height to be $H$. In terms of cylindrical coordinates, $C_1 = \{(r,\theta,z) \mid 0 \le r \le R, 0 \le \theta \le 2\pi, 0 \le z \le H\}$.

When rolling down a slope, only one axis of rotation makes sense, namely the z-axis. We therefore calculate the moment of inertia about the z-axis for $C_1$ with constant density $\rho$:

$$I_1 = \iiint_{C_1} (x^2+y^2)\rho \, dV = \int_0^H \int_0^R \int_0^{2\pi} \rho r^2 \, r \, d\theta \, dr \, dz$$

(note $x^2+y^2 = r^2$ for $x = r\cos\theta, y = r\sin\theta$)

$$= \rho \left(\int_0^R r^3 \, dr\right)\left(\int_0^H dz\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \rho \times \tfrac{1}{4}R^4 \times H \times 2\pi = \frac{\rho R^4 H \pi}{2}.$$

Spherical Shell : In terms of spherical coordinates,

$$S = \{(r,\theta,\phi) \mid a \le r \le R, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}.$$

Since density is constant, by symmetry, when the object is rolling down a slope, the axis of rotation is irrelevant when calculating the moment of inertia. We calculate the moment of inertia about the z-axis for $S$ with constant density $\rho$:

$$I_2 = \iiint_S (x^2+y^2)\rho \, dV = \rho \int_a^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin^2\phi \cdot r^2 \sin\phi \, d\phi \, d\theta \, dr$$

(note $x^2+y^2 = r^2\sin^2\phi$ for $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$)

$$= \rho \left(\int_a^R r^4 \, dr\right)\left(\int_0^{2\pi} d\theta\right)\underbrace{\left(\int_0^{\pi} (1-\cos^2\phi)\sin\phi \, d\phi\right)}_{(\text{set } u = -\cos\phi)}$$

$$= \rho \times \tfrac{1}{5}(R^5 - a^5) \times 2\pi \times \int_{-1}^{1} (1-u^2) \, du$$

$$= \tfrac{2}{5}\rho\pi(R^5 - a^5) \times \left[u - \tfrac{1}{3}u^3\right]_{-1}^{1}$$

$$= \tfrac{2}{5}\rho\pi(R^5 - a^5) \times \tfrac{4}{3} = \frac{8}{15}\rho\pi(R^5 - a^5)$$

Set $I_1 = I_2$

$$\Rightarrow \frac{1}{2}\rho\pi R^4 H = \frac{8}{15}\rho\pi(R^5 - a^5) \Rightarrow H = \frac{16}{15}\left(\frac{R^5 - a^5}{R^4}\right)$$

Q4. Cont'.

Cylindrical Shell  Take the height to be $\bar{H}$. In terms of cylindrical coords

$$C_2 = \{(r,\theta,z) \mid 0 \le z \le \bar{H},\ 0 \le \theta \le 2\pi,\ a \le r \le R\}.$$

As with $C_1$, take axis of rotation to be the z-axis.

$$\Rightarrow I_3 = \iiint_{C_2} (x^2+y^2)\rho \, dV = \rho \int_0^{\bar{H}} \int_a^R \int_0^{2\pi} r^2 \cdot r \, d\theta \, dr \, dz$$

$$= \rho \left( \int_a^R r^3 dr \right) \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\bar{H}} dz \right)$$

$$= \rho \times \frac{1}{4}(R^4 - a^4) \times 2\pi \times \bar{H}$$

$$= \frac{1}{2} \rho \pi (R^4 - a^4) \bar{H}.$$

Set $I_3 = I_2$

$$\Rightarrow \frac{1}{2} \rho \pi (R^4 - a^4) \bar{H} = \frac{8}{15} \rho \pi (R^5 - a^4)$$

$$\Rightarrow \bar{H} = \frac{16}{15} \left( \frac{R^5 - a^5}{R^4 - a^4} \right)$$

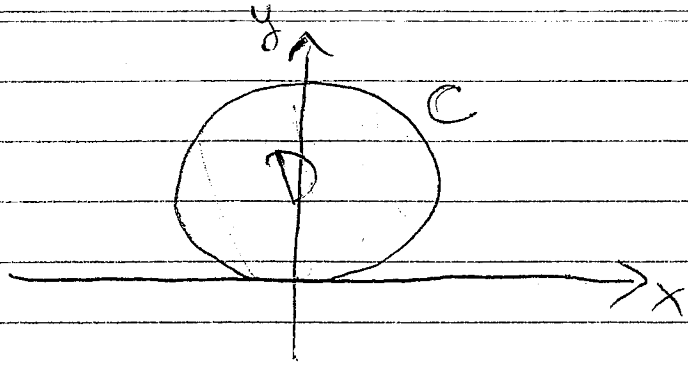
Compare with solid cylinder:

$$H = \frac{16}{15} \left( \frac{R^5 - a^5}{R^4} \right)$$

$$\Rightarrow H = \left( \frac{R^4 - a^4}{R^4} \right) \bar{H}.$$

$\Rightarrow$ cylindrical shell has larger height since $0 < \frac{R^4 - a^4}{R^4} < 1$ for $0 < a < R$.

Q5.




$\underline{F}(x,y) = x^2 \underline{i} + xy \underline{j}$

For $\oint_C \underline{F} \cdot \underline{T}\, ds$ : there are two approaches.

① Parametrise C : $\underline{r}(t) = \cos t\, \underline{i} + (\sin t + 1)\underline{j}$ , $0 \le t \le 2\pi$.
(which is anticlockwise around C)

$\underline{F}(t) = \cos^2 t\, \underline{i} + \cos t\,(1 + \sin t)\underline{j}$

$\underline{r}'(t) = -\sin t\, \underline{i} + \cos t\, \underline{j}$

$\Rightarrow \underline{F}(t) \cdot \underline{r}'(t) = -\cos^2 t \sin t + \cos^2 t\,(1 + \sin t) = \cos^2 t$.

$$\Rightarrow \oint_C \underline{F} \cdot \underline{T}\, ds = \int_0^{2\pi} \cos^2 t\, dt$$

$$= \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right) dt \quad (\text{trig. ident.})$$

$$= \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t \Big|_0^{2\pi} = \pi.$$

OR ② Use Green's theorem $\oint_C \underline{F} \cdot \underline{T}\, ds = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$.

In terms of "shifted" polar coordinates $x = r\cos\theta$, $y = 1 + r\sin\theta$,

$D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y = 1 + r\sin\theta$

$$\Rightarrow \oint_C \underline{F} \cdot \underline{T}\, ds = \int_0^1 \int_0^{2\pi} (1 + r\sin\theta)\, r\, d\theta\, dr$$

$$= \left(\int_0^1 r\, dr\right)\left(\int_0^{2\pi} d\theta\right) + \left(\int_0^1 r^2\, dr\right)\underbrace{\left(\int_0^{2\pi} \sin\theta\, d\theta\right)}_{0}$$

$$= \tfrac{1}{2} \times 2\pi = \pi.$$

Q5 Cont'' : For $\oint_C \underline{F}\cdot\underline{n}\, ds$ there are also two approaches.

① Use the same parametrisation as before for C :

$\underline{r}(t) = \cos t\,\underline{i} + (1+\sin t)\underline{j}$ , $0 \le t \le 2\pi$.

$\Rightarrow \underline{r}'(t) = -\sin t\,\underline{i} + \cos t\,\underline{j}$ , $|\underline{r}'(t)| = 1 \Rightarrow$ in $\oint_C$, $ds = |\underline{r}'(t)|\,dt = dt$.

$\longrightarrow \underline{r}'(t)$ is a unit tangent vector (anticlockwise around C)

$$\Rightarrow \underline{n} = \underline{r}'(t) \times \underline{k} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos t\,\underline{i} + \sin t\,\underline{j}$$

is a unit normal vector directed outward from the region enclosed by C.

$$\oint_C \underline{F}\cdot\underline{n}\, ds = \int_0^{2\pi} \left(\cos^2 t\,\underline{i} + \cos t\,(1+\sin t)\underline{j}\right)\cdot\left(\cos t\,\underline{i} + \sin t\,\underline{j}\right) dt.$$

$$= \int_0^{2\pi} \left(\cos^3 t + \cos t \sin t + \cos t \sin^2 t\right) dt.$$

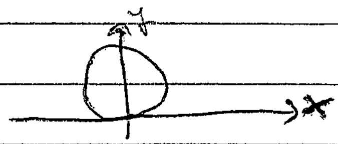
$$= 0 \quad \text{(each term has an odd power of } \cos t\text{)}$$

---

or ② Use the flux form of Green's theorem

$$\oint \underline{F}\cdot\underline{n}\, ds = \iint_D \nabla\cdot\underline{F}\, dA \qquad \left(\text{Note } \nabla\cdot\underline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) = 3x\right)$$

$$\Rightarrow \oint_C \underline{F}\cdot\underline{n}\, ds = \iint_D 3x\, dA$$




Since D is symmetric about the y-axis, interpreting the double integral as "net volume", and noting the "surface" $z = 3x$ has as much volume above the x-y plane (to the right of the y-axis) as below (to the left of the y-axis), we must have $\iint_D 3x\, dA = 0$.

( Can check directly using shifted polar coords $x = r\cos\theta$, $y = 1 + r\sin\theta$.

$$\iint_D 3x\, dA = 3\int_0^1 \int_0^{2\pi} r\cos\theta \cdot r\, d\theta\, dr = 0$$ ).