(1) Given $\underline{r}(t) = t\underline{i} + \cos\left(\frac{\pi}{2}t^2\right)\underline{j}$, $0 \le t \le 1$

note that $\underline{r}(0) = \underline{j}$ (start at $(0,1)$)

$\underline{r}(1) = \underline{i}$ (finish at $(1,0)$)

Also, $\underline{F}(x,y) = (1+2xy)\underline{i} + (1+x^2)\underline{j}$

$$F_1 = 1+2xy, \quad F_2 = 1+x^2$$

$$\Rightarrow \frac{\partial F_1}{\partial y} = 2x, \quad \frac{\partial F_2}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \Rightarrow \underline{F} \text{ is conservative}$$

$$\Rightarrow \exists\, \phi(x,y) \text{ s.t. } F_1 = \frac{\partial \phi}{\partial x}, \quad F_2 = \frac{\partial \phi}{\partial y}$$

Now $\frac{\partial \phi}{\partial x} = F_1 = 1+2xy \Rightarrow \phi(x,y) = x + x^2y + g(y)$.

$$\Rightarrow \frac{\partial \phi}{\partial y} = x^2 + g'(y)$$

$$= F_2 = 1+x^2$$

$\Rightarrow g'(y) = 1 \Rightarrow g(y) = y + C$ (constant $c$).

$\Rightarrow \phi(x,y) = x + y + x^2y$ is a potential function for $\underline{F}$ (up to constant).

The fundamental theorem of line integrals states that

$$\int_C \underline{F} \cdot d\underline{r} = \phi(\text{finish}) - \phi(\text{start})$$

$$= \phi(1,0) - \phi(0,1)$$

$$= 1 - 1 = 0.$$

(2) $\underline{F}(x,y,z) = 3xy^2\underline{i} + x\cos z\,\underline{j} + z^3\underline{k}$

Net outward flux $= \oiint_S \underline{F}\cdot\underline{n}\,dS$

$= \iiint_V \underline{\nabla}\cdot\underline{F}\,dV$ ...

By Gauss' divergence theorem, since all the conditions of the theorem are met by $S$ and $\underline{F}$ & $S$ is the surface of the solid cylinder $V$.

To describe $V$, use cylindrical coordinates, but rotated:

$$\left.\begin{array}{l} x \to y \\ y \to z \\ z \to x \end{array}\right\} \text{ so } \quad \begin{array}{l} y = r\cos\theta \\ z = r\sin\theta \\ x \end{array}$$

& Jacobian still $r$.

such that

$$V = \{(x, r, \theta) \mid -1 \le x \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1\}.$$

$$\underline{\nabla}\cdot\underline{F} = 3y^2 + 3z^2 = 3r^2.$$

$$\Rightarrow \text{flux} = \int_{-1}^{2}\int_{0}^{2\pi}\int_{0}^{1} 3r^2 \cdot r\,dr\,d\theta\,dx$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 3r^3\,dr\right)\left(\int_{-1}^{2} dx\right)$$

$$= 2\pi \times \frac{3}{4} \times (2-(-1)) = \frac{9}{2}\pi.$$

(3) $\oint\!\!\!\oint_S f(x,y,z)\,dS = \oint\!\!\!\oint_S g\left(\sqrt{x^2+y^2+z^2}\right) dS$

where $S$ is the surface of the sphere $x^2+y^2+z^2=9$.

i.e. in $\oint\!\!\!\oint_S$, we are only interested in points on $S$

$$\Rightarrow \oint\!\!\!\oint_S g\left(\sqrt{x^2+y^2+z^2}\right) dS = \oint\!\!\!\oint_S g\left(\sqrt{9}\right) dS$$

$$= \oint\!\!\!\oint_S g(3)\,dS$$

$$= \oint\!\!\!\oint_S (-2)\,dS \qquad \left(\begin{array}{l}\text{since } g(t)=t-5 \\ \Rightarrow g(3)=-2\end{array}\right).$$

$$= -2 \oint\!\!\!\oint_S dS$$

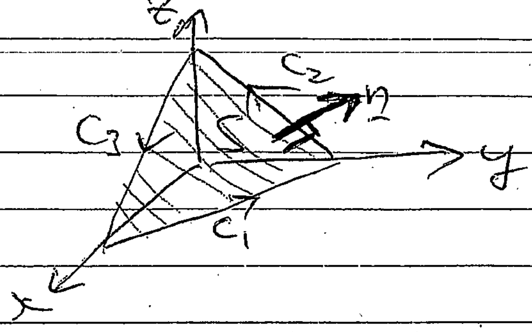
$$= -2 \times (\text{surface area of sphere})$$

$$= -2 \times 4 \times \pi \times 3^2 = -72\pi.$$

(4) Verify $\oint_C \underline{F} \cdot d\underline{r} = \iint_S (\nabla \times \underline{F}) \cdot \underline{n} \, dS$

for $F = x\underline{i} + y\underline{j} + xyz\underline{k}$.

& S and $C = C_1 \cup C_2 \cup C_3$ are indicated in the diagram




$C_1$ : $\underline{r}(t) = (1-t)\underline{i} + 2t\underline{j}$, $0 \le t \le 1$.

$\Rightarrow \underline{r}'(t) = -\underline{i} + 2\underline{j}$ & $\underline{F}(t) = (1-t)\underline{i} + 2t\underline{j}$

$$\int_{C_1} \underline{F} \cdot d\underline{r} = \int_0^1 \left( \underline{F}(t) \cdot \frac{d\underline{r}}{dt} \right) dt$$

$$= \int_0^1 \left( -(1-t) + 4t \right) dt = -1 + \frac{5}{2} = \frac{3}{2}.$$

$C_2$ : $\underline{r}(t) = 2(1-t)\underline{j} + 2t\underline{k}$, $0 \le t \le 1$

$\Rightarrow \underline{r}'(t) = -2\underline{j} + 2\underline{k}$ & $\underline{F}(t) = 2(1-t)\underline{j}$

$$\Rightarrow \int_{C_2} \underline{F} \cdot d\underline{r} = \int_0^1 \left( \underline{F}(t) \cdot \underline{r}'(t) \right) dt$$

$$= \int_0^1 -4(1-t) \, dt = -4 + 2 = -2.$$

$C_3$ : $\underline{r}(t) = t\underline{i} + 2(1-t)\underline{k}$.

$\Rightarrow \underline{r}'(t) = \underline{i} - 2\underline{k}$ & $\underline{F}(t) = t\underline{i}$

$$\Rightarrow \int_{C_3} \underline{F} \cdot d\underline{r} = \int_0^1 \left( \underline{F}(t) \cdot \underline{r}'(t) \right) dt$$

$$= \int_0^1 t \, dt = \frac{1}{2}. \quad \oint_C \underline{F} \cdot d\underline{r} = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$\Rightarrow \oint_C \underline{F} \cdot d\underline{r} = \frac{3}{2} - 2 + \frac{1}{2} = 0.$$

Now $\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & xyz \end{vmatrix} = xz\underline{i} - yz\underline{j}$

~~Use $\underline{r}(u,v) = u\underline{i} + v\underline{j}$~~

Use $\underline{r}(u,v) = (1-u)\underline{i} + 2u(1-v)\underline{j} + 2uv\underline{k}$, $0 \le u \le 1$, $0 \le v \le 1$.

$\underline{r}_u = -\underline{i} + 2(1-v)\underline{j} + 2v\underline{k}$

$\underline{r}_v = -2u\underline{j} + 2u\underline{k}$.

$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 2-2v & 2v \\ 0 & -2u & 2u \end{vmatrix} = 4u\underline{i} + 2u\underline{j} + 2u\underline{k}$.

(direction ok since consistent with orientation of S i.e. has positive $\underline{k}$ component)

& $\nabla \times F(u,v) = 2u(1-u)v\underline{i} - 4u^2v(1-v)\underline{j}$

$$\Rightarrow (\nabla \times \underline{F}(u,v)) \cdot (\underline{r}_u \times \underline{r}_v) = 8u^2(1-u)v - 8u^3v(1-v)$$
$$= 8u^2v - 16u^3v + 8u^3v^2$$

$$\Rightarrow \iint_S (\nabla \times \underline{F}) \cdot \underline{n}\, dS = \int_0^1 \int_0^1 (8u^2v - 16u^3v + 8u^3v^2)\, du\, dv$$

$$= 8 \times \frac{1}{3} \times \frac{1}{2} - 16 \times \frac{1}{4} \times \frac{1}{2} + 8 \times \frac{1}{4} \times \frac{1}{3}$$

$$= \frac{4}{3} - 2 + \frac{2}{3} = 0.$$

Hence $\oint_C \underline{F} \cdot d\underline{r} = 0 = \iint_S (\text{curl}\, \underline{F}) \cdot \underline{n}\, dS$

in this case.

(5) $A = \begin{pmatrix} 0 & 1 & 1 & -3 \\ 2 & 6 & 2 & 9 \\ 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 4 \end{pmatrix}$ $R_1 \leftrightarrow R_4$

$\longrightarrow \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 6 & 2 & 9 \\ 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 \end{pmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\longrightarrow \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 0 & 0 & 1 \\ 3 & -8 & -3 & -12 \\ 0 & 1 & 1 & -3 \end{pmatrix}$ $R_2 \leftrightarrow R_4$

$\longrightarrow \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 3 & -8 & -3 & -12 \\ 2 & 0 & 0 & 1 \end{pmatrix}$ $R_3 \to R_3 - (-8)R_2$

$\longrightarrow \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 3 & -8 & 5 & -36 \\ 2 & 0 & 0 & 1 \end{pmatrix}$

Here we use the notation (n) in the $(i,j)$ entry means the row operation $R_i \to R_i - nR_j$ was used to make *that* entry zero (introduced in lectures). This method does not have to be used, as long as $L$ summarises the row operations (type 2 only) used.

$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -8 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 5 & -36 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

so that $P_{24}P_{14}A = LU$

$\Rightarrow A = P_{14}P_{24}LU = PLU$

where

$P = P_{14}P_{24} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$