MATH2000

CALCULUS
AND
LINEAR ALGEBRA II

Lecture Workbook

Summer Semester 2010-2011

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1 Solutions of first order ODEs

By the end of this section, you should be able to answer the following questions about first order ODEs:

- How do you solve an IVP associated with directly integrable, separable or linear ODEs? (Revision)
- Under what conditions does a solution to an IVP problem exist?
- Under what conditions is a solution to an IVP problem unique?

In MATH1052, you were introduced to Ordinary Differential Equations (ODEs) and Initial Value Problems (IVPs) and saw how to find solutions to some special types of first order equations. In particular, there should be three types of first order ODEs that you are familiar with solving.

- Directly integrable: $\dfrac{dy}{dx} = f(x)$.

$$\Longrightarrow \quad y(x) = \int f(x)\,dx + \text{const.}$$

- Separable: $\dfrac{dy}{dx} = f(x)g(y)$.

$$\Longrightarrow \int \frac{1}{g(y)} \frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\Rightarrow \int \frac{1}{g(y)}\,dy = \int f(x)\,dx + \text{const.}$$

- Linear: $\dfrac{dy}{dx} = q(x) - p(x)y$.

$$I\frac{dy}{dx} + Ip(x)y = Iq(x)$$

s.t. $\dfrac{d}{dx}(Iy) = Iq$. & integrate.

($I = e^{\int p\,dx}$ is called an "integrating factor")

In most applications involving first order ODEs, we are required to solve an IVP. Generally, this is a problem of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

In otherwords, we seek to find solutions of the ODE which pass through the point (x_0, y_0) in the x-y plane.

Consider the following three examples.

1.1 Example: $\frac{dy}{dx} = x$, $y(0) = 1$ has a unique solution

$\Longrightarrow \quad y(x) = \frac{1}{2}x^2 + c$

$y(0) = 1 \quad \Longrightarrow \quad 1 = c$

$\Longrightarrow \quad y = \frac{1}{2}x^2 + 1$

real cube root.

1.2 Example: $\frac{dy}{dx} = 3xy^{1/3}$, $y(0) = 0$ has more than one solution

$\Longrightarrow \quad \int y^{-1/3} \frac{dy}{\cancel{dx}} \cancel{dx} = \int 3x \, dx$

$\Longrightarrow \quad \frac{3}{2} y^{2/3} = \frac{3}{2} x^2 + c$

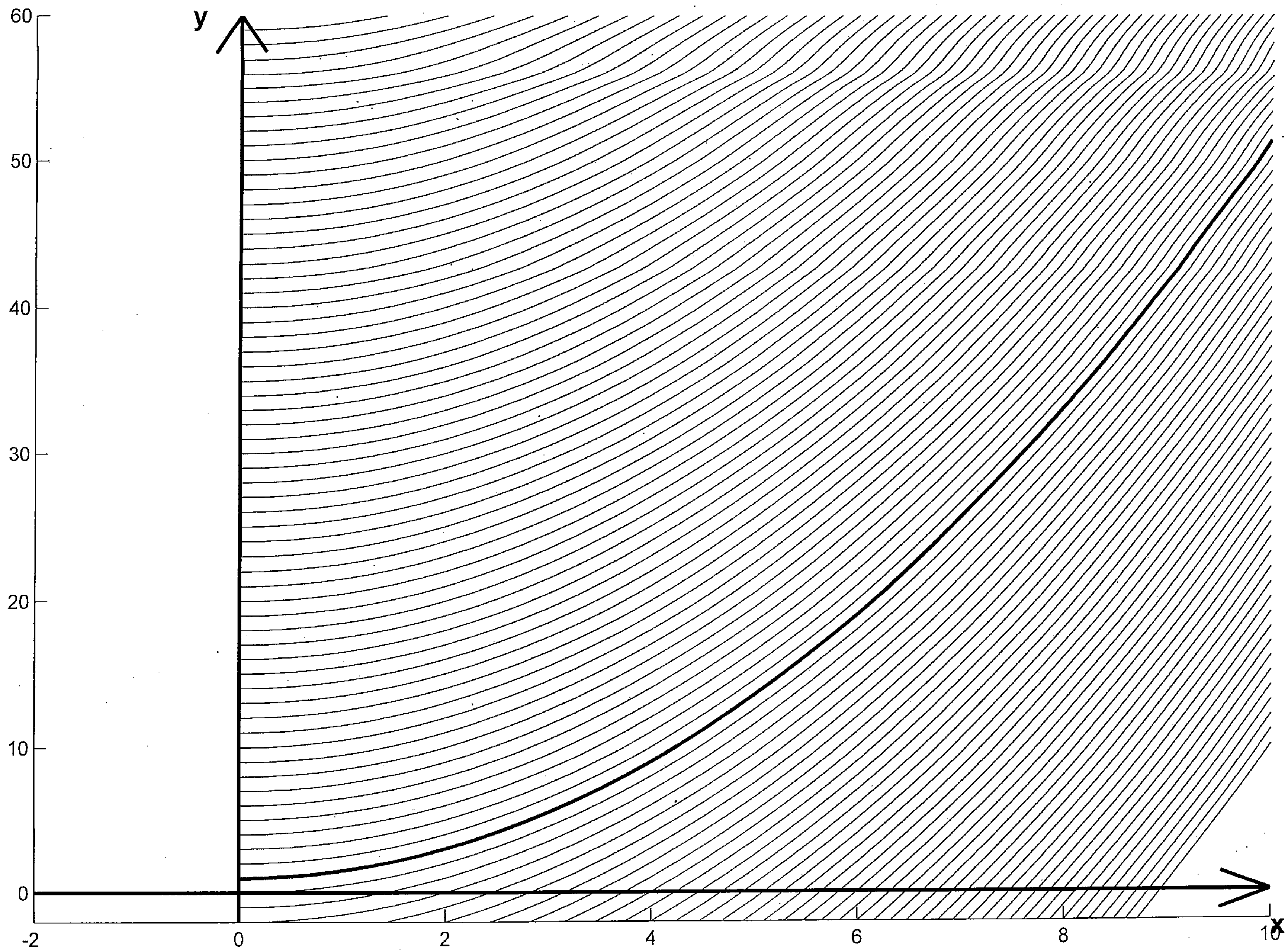
$x = 0, \; y = 0 \quad \Longrightarrow \quad c = 0$

$\Longrightarrow \quad y^{2/3} = x^2 \quad \Longrightarrow \quad \underline{y = x^3} \text{ or } \underline{y = -x^3}$

Also, $y = 0$ is an equilibrium solution.

In fact, $y = \begin{cases} 0 & x \leq k \quad (k > 0) \\ \pm\sqrt{(x^2 - k^2)^3} & x > k. \end{cases}$

$\Longrightarrow$ infinitely many solutions.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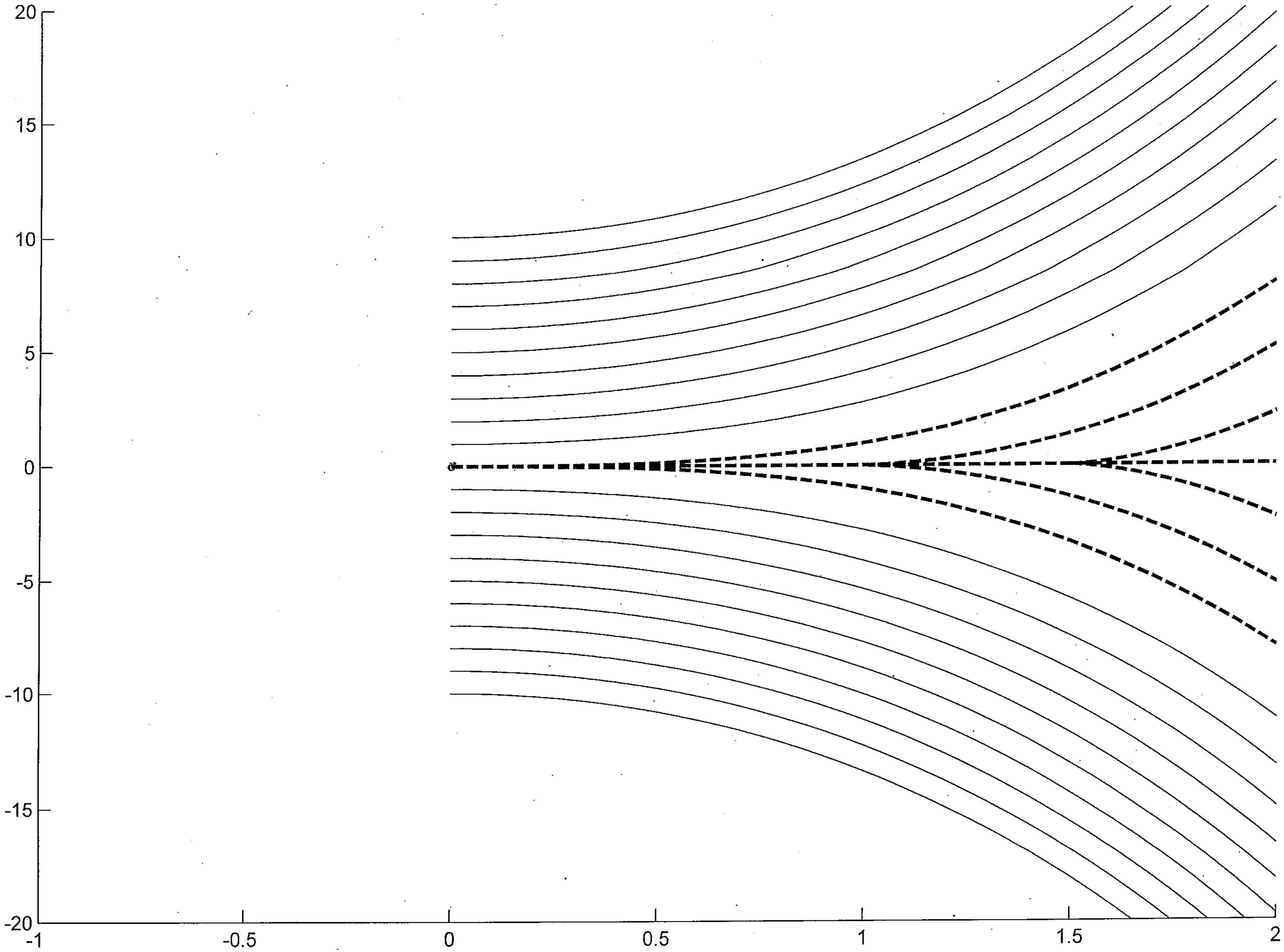
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1.3 Example: $\dfrac{dy}{dx} = \dfrac{x-y}{x}$, $y(0) = 1$ has no solution

$\Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x} y = 1$

mult. by x (integrating factor)

$\Rightarrow \left(x \dfrac{dy}{dx} + y\right) = x$

$\dfrac{d}{dx}(xy) = x$

$\Rightarrow xy = \frac{1}{2}x^2 + c$

(family of hyperbolae!? → see lect. 3.4.)

$\Rightarrow$ no solution satisfying $y(0) = 1$.

1.4 Existence and uniqueness criteria

THEOREM:

Here we consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

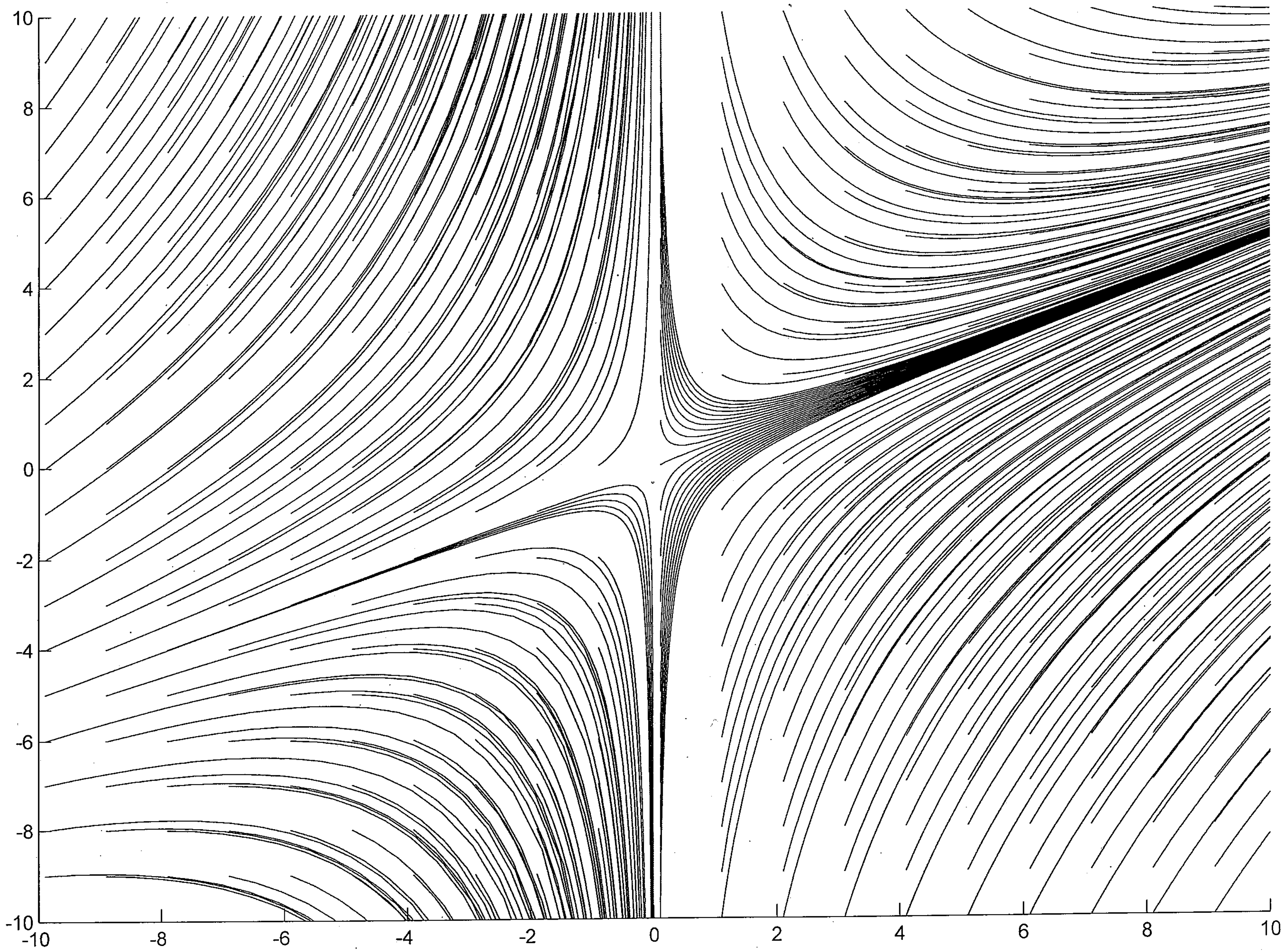
- (existence) If $f(x, y)$ is continuous in some rectangle

$$R = \{(x, y)|\ |x - x_0| < a, |y - y_0| < b\}$$

  then the initial value problem has at least one solution. Note that $(x_0, y_0) \in R$.

- (uniqueness) Moreover, if $f_y(x, y)$ is also continuous in R then there is at most one solution to the initial value problem.

The above two conditions only tell us that a solution exists or is unique locally (i.e., in the rectangle R). Beyond R, we simply don't know. Let's look at the previous three examples in the context of the theorem.



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1.5 Example: $\dfrac{dy}{dx} = x$, $y(0) = 1$

$f(x,y) = x$ cts $\Longrightarrow$ existence

$f_y = 0$ cts $\Longrightarrow$ uniqueness

1.6 Example: $\dfrac{dy}{dx} = 3xy^{1/3}$, $y(0) = 0$

$f(x,y) = 3xy^{1/3}$ cts $\Longrightarrow$ existence

$f_y = \dfrac{x}{y^{2/3}}$ not cts in any rectangle containing $(0,0)$

$\Longrightarrow$ cannot say that there is a unique solution.

1.7 Example: $\dfrac{dy}{dx} = \dfrac{x-y}{x}$, $y(0) = 1$

$f(x,y) = \dfrac{x-y}{x}$ not cts in any rectangle containing $(0,1)$

$\Longrightarrow$ cannot say solutions exist.