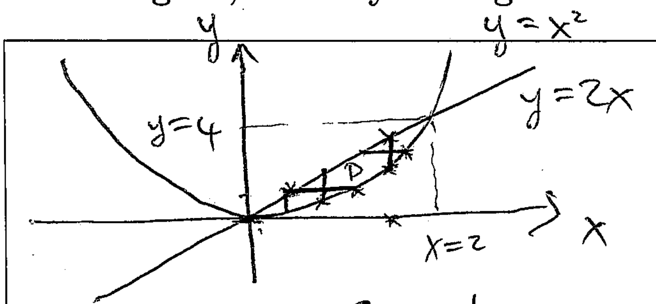
10 Interchanging order of integration

By the end of this section, you should be able to answer the following questions:

- How do you change the order of integration in a double integral?
- when might it be necessary to change the order of integration in a double integral?

It is often possible to represent a type I region as a union of type II regions, or a type II region as a union of type I regions. Why would we want to do that? In some cases, it may only be possible to integrate a function one way but not the other. In this section, we investigate this idea more closely.

10.1 Find the volume under the paraboloid $z = x^2 + y^2$ above the region D, where D is bounded by $y = x^2$ and $y = 2x$. Do the problem twice, first by taking D to be a type I region, then by taking D to be type II.



Points of intersection

$x^2 = 2x$

$\Rightarrow x = 0, 2$

Type I: $D = \{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$

$$\Rightarrow \text{vol.} = \iint_D (x^2+y^2)\,dA$$

$$= \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^2 \left[x^2 y + \frac{1}{3}y^3\right]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \left(2x^3 + \frac{8}{3}x^3 - \left(x^4 + \frac{1}{3}x^6\right)\right) dx$$

$$= \ldots = \frac{216}{35}$$

$$y = x^2 \Rightarrow x = \pm\sqrt{y}$$

$$\boxed{\begin{aligned}
&\text{Type II}: D = \{(x,y) \mid 0 \le y \le 4,\ \tfrac{y}{2} \le x \le \sqrt{y}\}. \\
&\Longrightarrow \text{vol.} = \iint_D (x^2+y^2)\,dA \\
&= \int_0^4 \left(\int_{y/2}^{\sqrt{y}} (x^2+y^2)\,dx \right) dy \\
&= \int_0^4 \left[\tfrac{1}{3}x^3 + xy^2 \right]_{x=y/2}^{x=\sqrt{y}} dy \\
&= \int_0^4 \left(\tfrac{1}{3} y^{3/2} + y^2 \cdot y^{\frac{1}{2}} - \left(\tfrac{1}{3}\left(\tfrac{y}{2}\right)^3 + \tfrac{y}{2}\cdot y^2 \right) \right) dy. \\
&= \ldots = \frac{216}{35}
\end{aligned}}$$

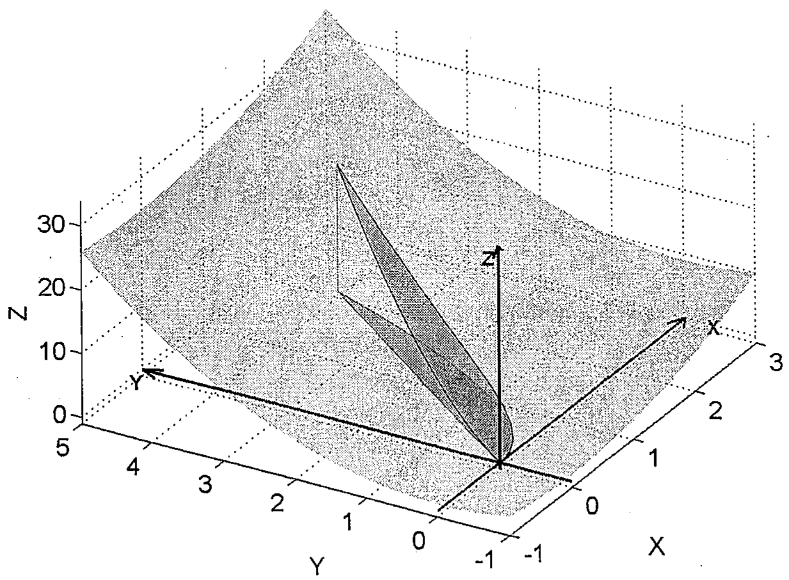


Figure 23: This volume can be calculated by treating the region in the x-y plane as either type I or II as seen in example 10.1.

MAIN POINT : cannot integrate one way ?

$\longrightarrow$ try changing order of integration.

In the following example, we see how it is sometimes necessary to change the order of integration in order to evaluate the integral.

10.2 Example: Find $\displaystyle\int_0^1 \left(\int_x^1 \sin(y^2)\, dy \right) dx$

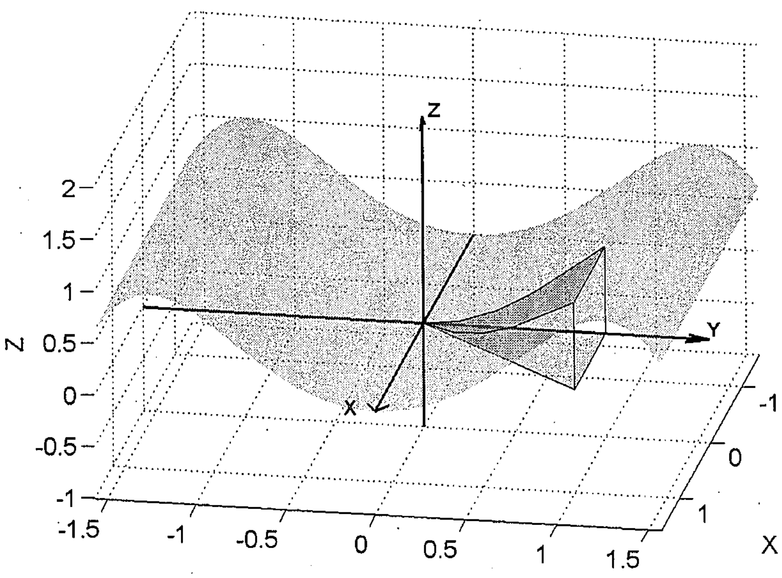


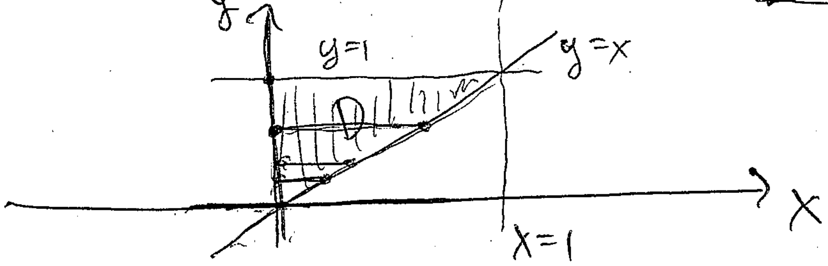
Figure 24: The volume described in example 10.2.

Stewart p 523: "Can we integrate all continuous functions?"

discussion on <u>elementary</u> <u>functions</u>

$\int \sin(y^2)\, dy$, $\int e^{-x^2} dx$.

$D = \{(x,y) \mid 0 \le x \le 1, \quad \underline{x \le y \le 1}\}.$



type II : $D = \{(x,y) \mid 0 \le y \le 1, \quad 0 \le x \le y\}.$

$\rightarrow$ integral $= \int_0^1 \left(\int_0^y \sin(y^2)\, dx \right) dy$

$$= \int_0^1 \Big[x \sin(y^2) \Big]_{x=0}^{x=y} dy.$$

$$= \int_0^1 \left(y \sin(y^2) - 0 \right) dy.$$

(set $u = y^2 \quad \frac{du}{dy} = 2y$.)

$\cdots \Rightarrow \quad -\frac{1}{2} \cos(y^2) \Big|_0^1$

$$= -\frac{1}{2}\cos(1) - \left(-\frac{1}{2}\cos(0)\right)$$

$$= \frac{1}{2}(1 - \cos(1))$$

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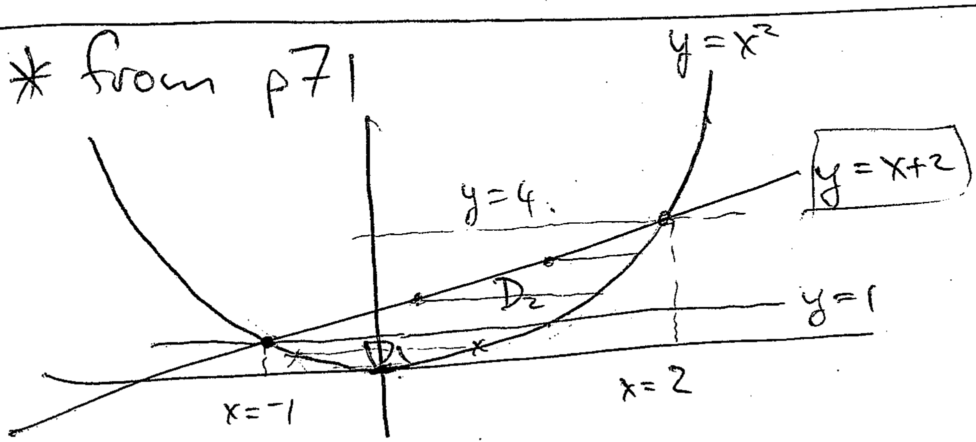
* go to p63.

$$= \int_{-1}^{2} \left[4xy + 5y^2 \right]_{y=x^2}^{y=x+2} dx$$

$$= \int_{-1}^{2} \left(4x(x+2) + 5(x+2)^2 - (4x^3 + 5x^4) \right) dx$$

$$= \ldots = 81$$

$*$ from p71



Type II ?

$D_1 = \{(x,y) \mid 0 \le y \le 1, -\sqrt{y} \le x \le \sqrt{y}\}$.

$D_2 = \{(x,y) \mid 1 \le y \le 4, y-2 \le x \le \sqrt{y}\}$.

$$\iint_D = \iint_{D_1} + \iint_{D_2}$$

TUTORIAL QUIZ 1

- ~~firsto~~ first 30 minutes under exam conditions
- next 20 minutes tutor goes over solutions.
- no calculators or other materials

3 Questions
1. ODES
2. Hyperbolic functions
3. Double integrals.

Workbook chap's 1-10.