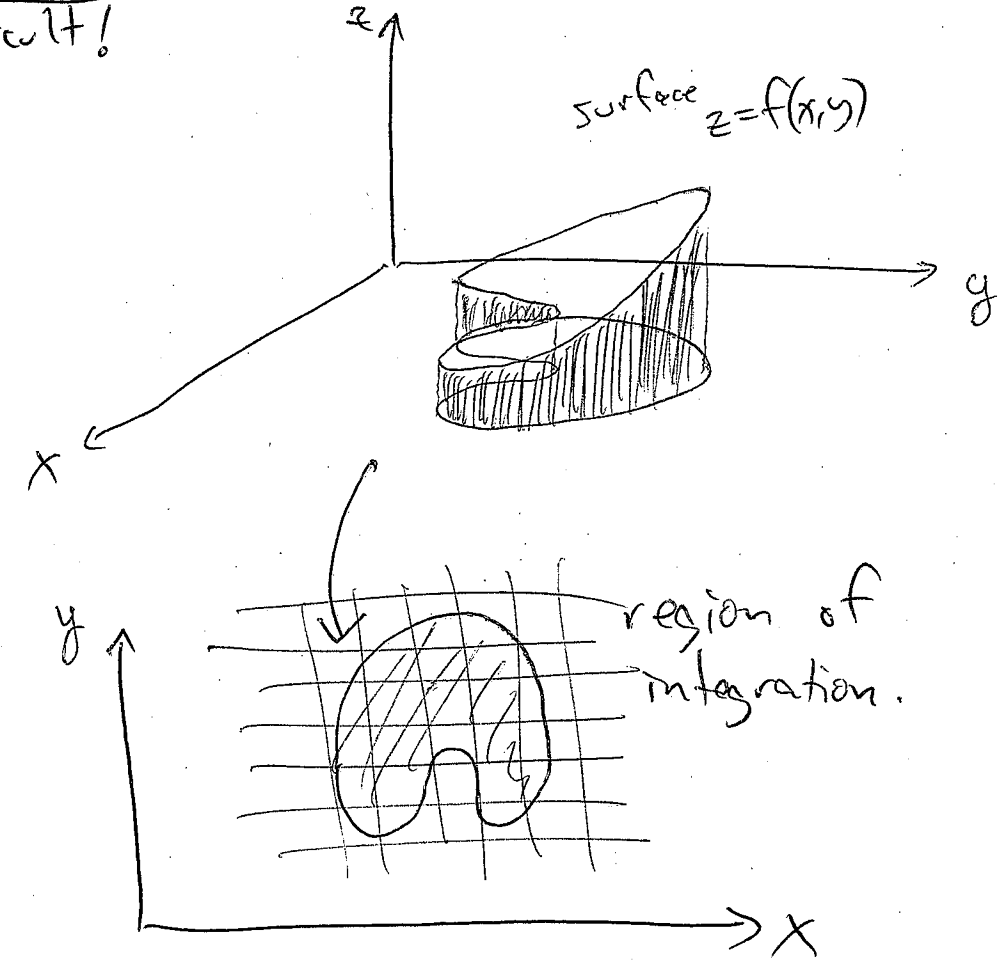
11 Review of applications: volume, area

Main points:

- This section is a review of applications of the double integral such as calculating net volume and area in the plane.
- By this stage you should be comfortable with using a double integral to calculate the net volume below a surface.
- You should know how to find the area of a general region in the plane.

When the regions are more difficult, it is a good idea to draw two diagrams - the 3-D diagram with the x-y-z axes and the 2-D one of the region in the x-y plane.

difficult!



$$\text{vol} \approx \sum \text{height} \times \Delta A \longrightarrow \iint_D f(x,y)\,dA.$$

$$\text{area} \approx \sum \Delta A \longrightarrow \iint_D dA.$$

11.1 Example: Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

y-z plane x-y plane.

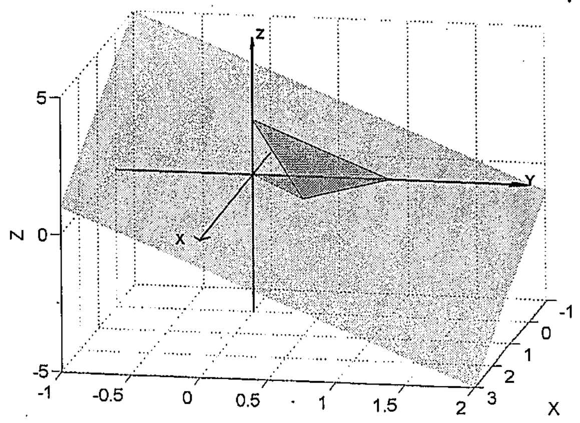
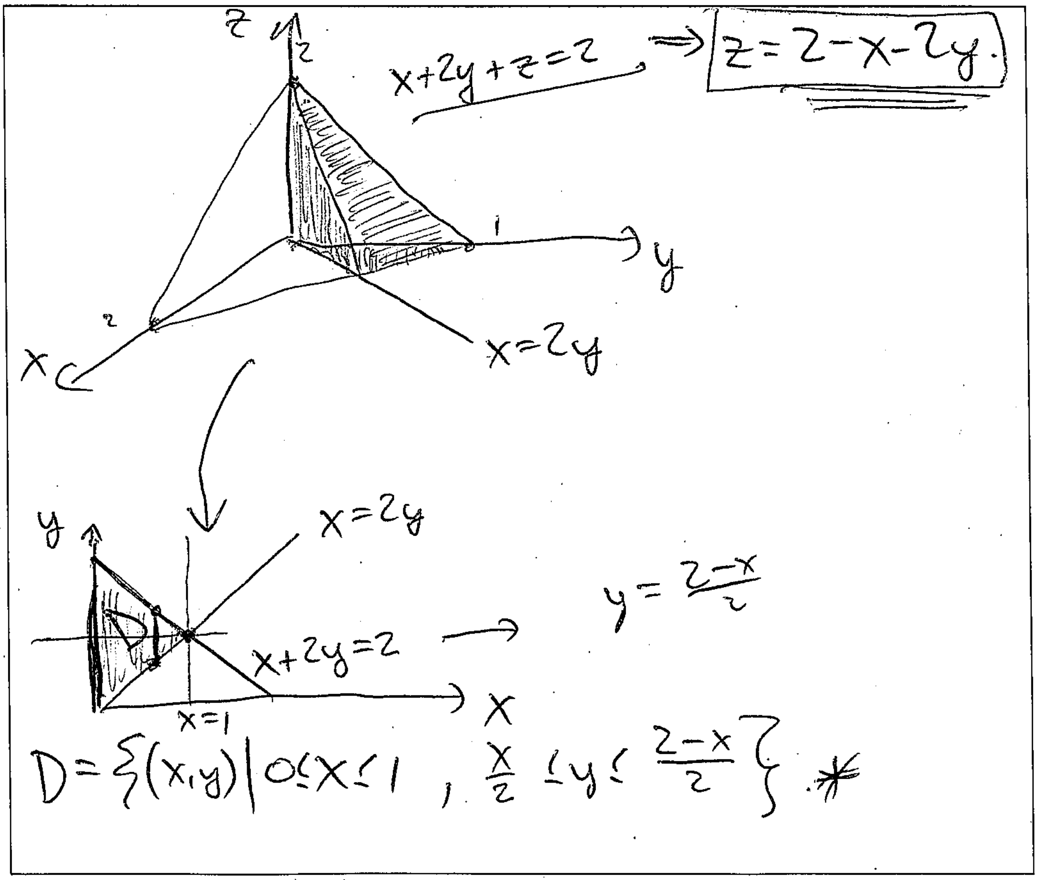


Figure 25: You should be able to reproduce a diagram like this one as an aid to determining the bounds of integration.



$$\text{Vol.} = \iint_D (2-x-2y)\, dA.$$

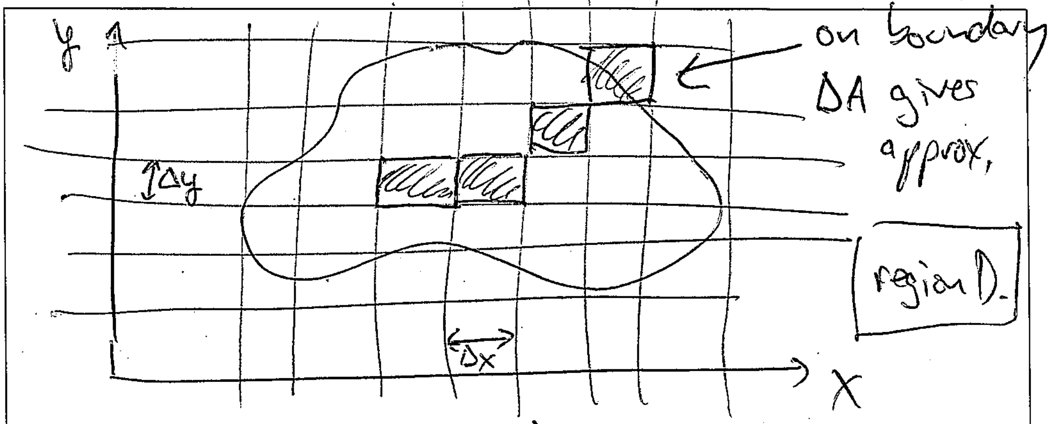
$$= \int_0^1 \left(\int_{\frac{x}{2}}^{\frac{2-x}{2}} (2-x-2y)\, dy \right) dx$$

$$= \int_0^1 \left[2y - xy - y^2 \right]_{y=\frac{x}{2}}^{y=\frac{2-x}{2}} dx.$$

$$= \dots = \tfrac{1}{3}.$$

11.2 Area

Note that if we take $f(x, y) = 1$, we have $\iint_D 1\, dA =$ area of the region D.



area $\Delta A = \Delta x \Delta y$.

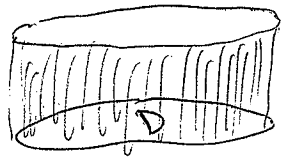
$\Rightarrow$ area of region $\approx \sum \Delta A \simeq \sum \Delta x \Delta y$

not exact on boundary.

obtain better approx. with smaller grid size, In limit $\Delta A \to 0$ $(\Delta x, \Delta y \to 0)$

$\Rightarrow$ area $= \lim\limits_{\Delta x, \Delta y \to 0} \sum \Delta x \Delta y$

$$= \iint_D dA.$$

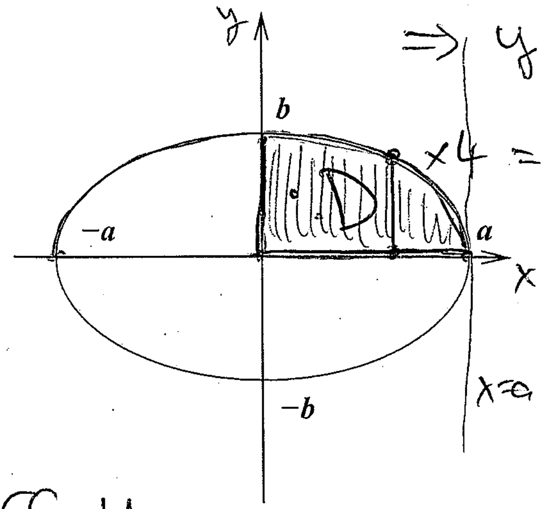


$z=1$ vol. below $z=1$ & above D.

$= \iint_D 1\, dA.$

11.3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\Rightarrow y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$



$\times 4$ = area of whole region.

area $= \iint_D dA$.

$$D = \left\{(x,y) \,\middle|\, 0 \le x \le a,\ 0 \le y \le b\sqrt{1-\frac{x^2}{a^2}}\right\}.$$

$\Rightarrow$ area of D $= \displaystyle\int_0^a \left(\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} 1\,dy \right) dx$

$$= \int_0^a \Big[\, y \,\Big]_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\, dx.$$

(subst. $x = a\sin\theta \ldots\ldots$)
$dx = a\cos\theta\, d\theta$ $\int \ldots$

$= \ldots = \dfrac{\pi ab}{4} \Rightarrow$ area of ellipse $= \pi ab$.