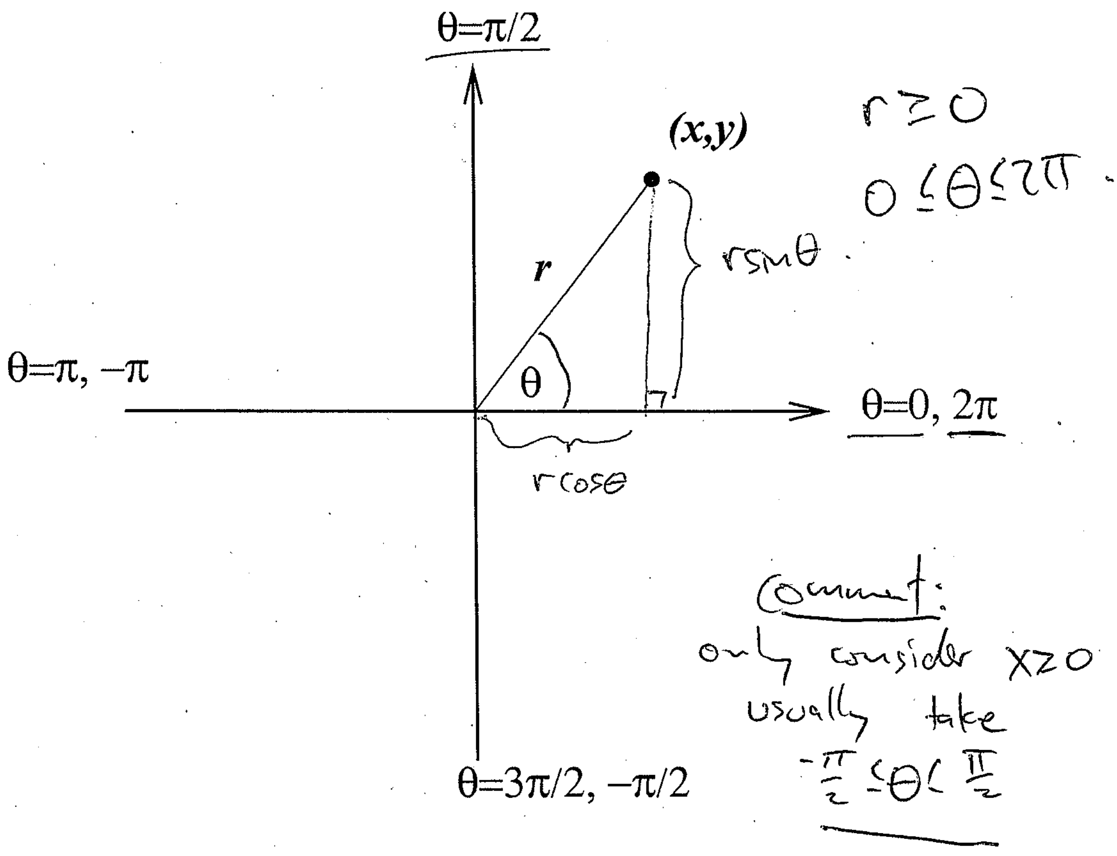
12 Double integrals in polar coordinates

By the end of this section, you should be able to answer the following questions:

- What is the relationship between polar coordinates and rectangular coordinates?
- How do you transform a double integral in rectangular coordinates into one in terms of polar coordinates?
- What is the Jacobian of the transformation?

For annular regions with circular symmetry, rectangular coordinates are difficult. It can be more convenient to use *polar coordinates.*

The following diagram explains the relationship between the polar variables r, θ and the usual rectangular ones x, y.



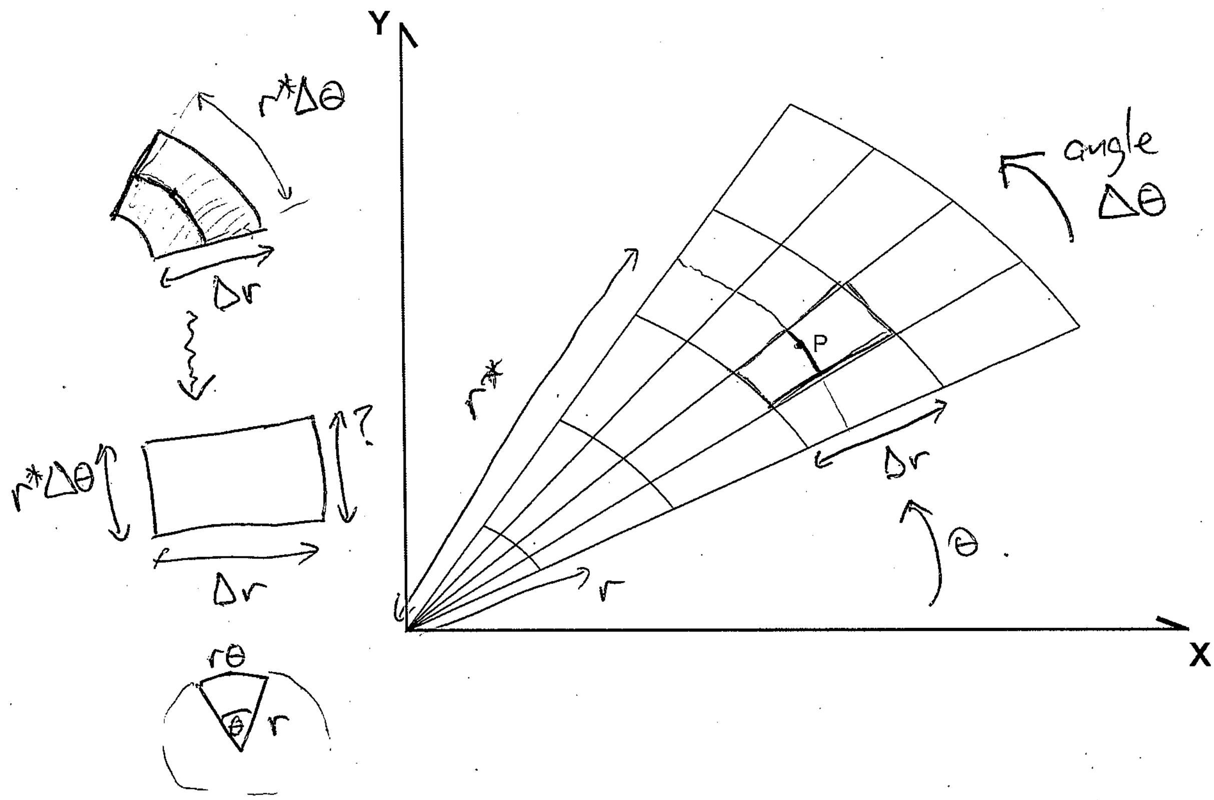
For polar coordinates, we ~~have~~

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta.}$$

$f(x,y) \geq 0$

Consider the volume of a solid beneath a surface $\underline{z = f(x,y)}$ and above a circular region in the x-y plane.

We divide the region into a $\boxed{\text{polar grid}}$ as in the following diagram:



We first approximate the area of each polar rectangle as a regular rectangle. We do this as follows. Choose a point P inside each polar rectangle in the polar grid. Let $P = (x^*, y^*)$ or in polar coordinates $\boxed{P = (r^*, \theta^*)}$, where

$$\underline{x^* = r^* \cos\theta^*, \quad y^* = r^* \sin\theta^*.}$$

The area of the polar rectangle containing P can be approximated as $r^*\Delta\theta\Delta r$. Therefore the volume under the surface and above each polar rectangle can be approximated as

$$\text{vol. one box} \approx \overbrace{\boxed{r^*\Delta\theta\Delta r}}^{\text{base area}} \underbrace{f(r^*\cos\theta^*, r^*\sin\theta^*)}_{\text{height}}.$$

Here $f(r^*\cos\theta^*, r^*\sin\theta^*)$ is the value of the function at the point P, which is also the height of the box used in the approximation. To obtain an approximation for

"Jacobian" of the variable transformation.

$\boxed{r}\,d\theta\, dr$

$x = x(u,v)$
$y = y(u,v)$

$$\Rightarrow \text{Jac.} = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

the entire volume below the surface, we sum over the entire polar grid:

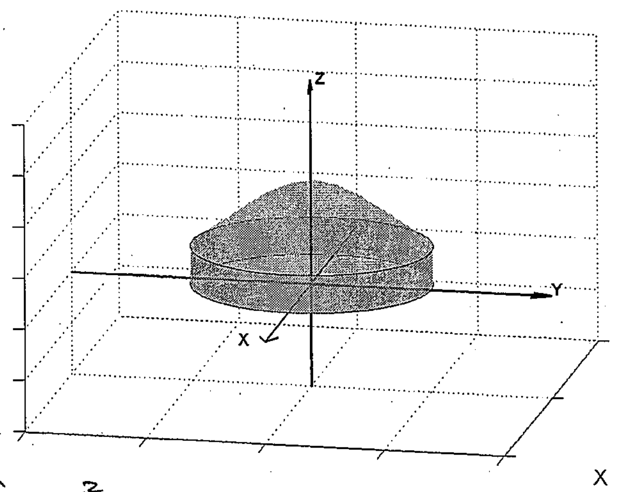
$$\begin{aligned} \text{vol.} &\approx \sum_{\text{(polar grid)}} \boxed{r^* \Delta\theta \Delta r} \underbrace{f(r^* \cos\theta^*, r^* \sin\theta^*)}, \\ \Rightarrow \quad \text{vol.} &= \lim_{\Delta r, \Delta\theta \to 0} \sum_{\text{(polar grid)}} r^* \Delta\theta \Delta r f(r^* \cos\theta^*, r^* \sin\theta^*) \\ &= \iint_D f(r\cos\theta, r\sin\theta) \underbrace{r\ d\theta\ dr}. \end{aligned}$$

The double integral in rectangular coordinates is then transformed as follows:

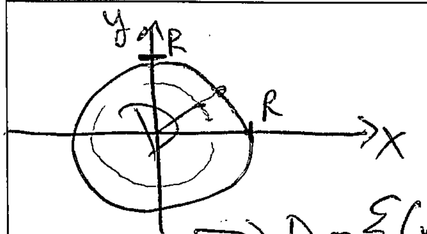
$$\boxed{\iint_R f(x,y) \underbrace{dx\ dy} = \iint_S f(r\cos\theta, r\sin\theta) \underbrace{r\ dr\ d\theta}.}$$

$dx\,dy \longrightarrow r\,d\theta\,dr.$

12.1 Example: Find $\iint_D \underbrace{e^{-(x^2+y^2)}}\ dx\ dy$ where D is the region bounded by the circle $x^2 + y^2 = R^2$.



Note $\int e^{-x^2} dx$ is NOT an elementary function.



Use polar coordinates.

$x = r\cos\theta, \quad y = r\sin\theta.$

$\Rightarrow x^2 + y^2 = r^2$

$\Rightarrow D = \{(r,\theta) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq R\}.$

Also, in $\iint_D$, $\underline{dx\,dy} \longrightarrow \underline{r\,d\theta\,dr}$.

$$\Rightarrow \iint_D = \int_0^R \int_0^{2\pi} e^{-r^2}\, r\, d\theta\, dr$$

$$(\text{prop. p59}) = \underbrace{\left(\int_0^R r e^{-r^2} dr\right)}_{\substack{\text{set } u=r^2 \\ \text{in} \int \; du = 2r\,dr}} \underbrace{\left(\int_0^{2\pi} d\theta\right)}_{= 2\pi}$$

$$= \ldots = 2\pi \times \frac{1}{2}\left(1-e^{-R^2}\right)$$

MORE FUN... Let $\underline{R \to \infty}$.

then $\iint_D$ represents vol. below $z = e^{-(x^2+y^2)}$

& above entire x-y plane.

$$\Rightarrow \iint_D = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2}\, dr\, d\theta = \lim_{R\to\infty} \pi\left(1-e^{-R^2}\right)$$

$$= \pi$$

(a finite value).

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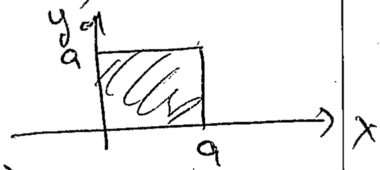
there's more... vol. above $x, y \geq 0$ quadrant $= \pi/4$.

since $z = e^{-(x^2+y^2)}$ is radially symmetric.

Consider $\displaystyle\int_0^a\int_0^a e^{-(x^2+y^2)}\,dx\,dy$



$$= \left(\int_0^a e^{-x^2}dx\right)\left(\int_0^a e^{-y^2}dy\right)$$

$$= \left(\int_0^a e^{-t^2}dt\right)^2$$

"dummy variable".

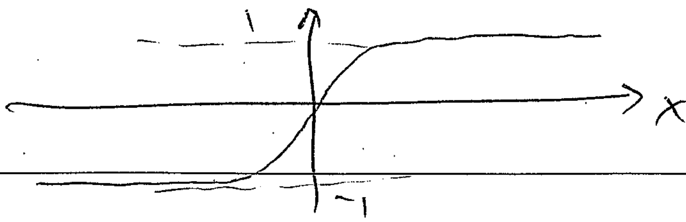
Let $a \to \infty$ (so consider vol. above entire quadrant)

$$\Rightarrow \left(\int_0^\infty e^{-t^2}dt\right)^2 = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$$

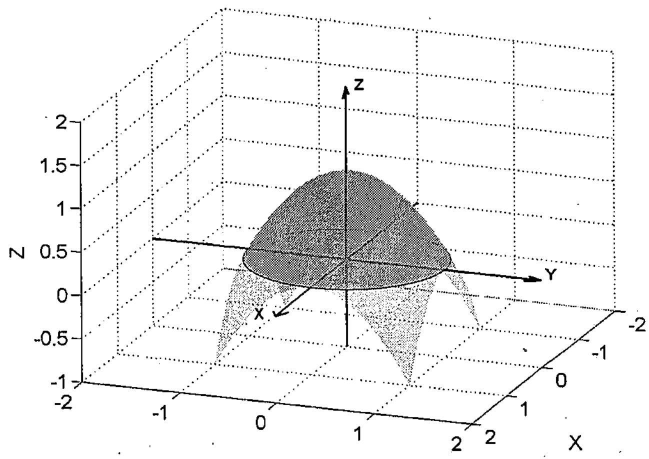
The "error function"

$$\mathrm{erf}(x) = \underbrace{\frac{2}{\sqrt{\pi}}} \int_0^x e^{-t^2}dt.$$



12.2 Example: Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

x-y plane.



surface $z = 1 - x^2 - y^2$ intersects x-y plane when $x^2 + y^2 = 1$

$x = r\cos\theta$, $y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$

$D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi.\}$.

$$\Rightarrow \text{vol.} = \iint_D (1 - x^2 - y^2)\, dA.$$

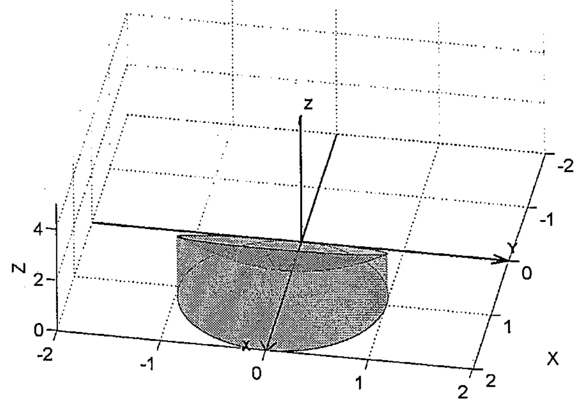
$$= \int_0^1 \int_0^{2\pi} (1 - r^2)\, r \, d\theta \, dr$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (r - r^3)\, dr\right)$$

$$= \cdots = \pi/2,$$

12.3 **Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and inside the cylinder $x^2 + y^2 = 2x$, for $z \geq 0$.**

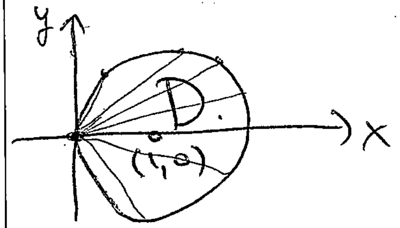
i.e. above circle in x-y plane.



$$x^2 + y^2 = 2x \Rightarrow (x^2 - 2x + 1) + y^2 = 0 + 1$$

$$\Rightarrow (x-1)^2 + y^2 = 1$$

circle of radius 1, centred at ~~x~~ $x = 1$, $y = 0$.



Could use $x = r\cos\theta$, $y = r\sin\theta$

try it! → complicated bounds.

(p 1014 Stewart)

$$x - 1 = r\cos\theta, \quad y = r\sin\theta.$$

$$\Rightarrow x = 1 + r\cos\theta.$$

"shifted" polar coordinates Jac. = r

$$z = x^2 + y^2 = (1 + r\cos\theta)^2 + r^2\sin^2\theta$$

$$= 1 + 2r\cos\theta + r^2(\cos^2\theta + \sin^2\theta)$$

$$= 1 + 2r\cos\theta + r^2$$

$D = \{(r,\theta) \mid 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}.$

$$\text{vol.} = \iint_D \underline{(x^2+y^2)}\, dA$$

Jacobian.

$$= \int_0^1 \int_0^{2\pi} (1 + 2r\cos\theta + r^2)\, r \, d\theta\, dr$$

$$(\text{prop. } 59) = \left(\int_0^1 r\, dr\right)\left(\int_0^{2\pi} d\theta\right) + \left(2\int_0^1 r^2 dr\right)\underbrace{\left(\int_0^{2\pi} \cancel{\cos\theta\, d\theta}\right)}_{0}$$

$$+ \left(\int_0^1 r^3 dr\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \frac{1}{2} \times 2\pi + \frac{1}{4} \times 2\pi = \frac{3}{2}\pi.$$