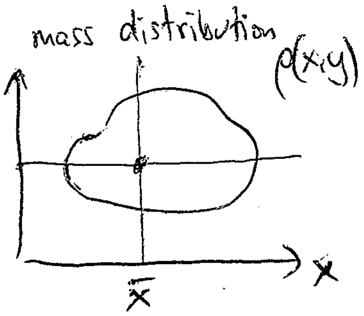
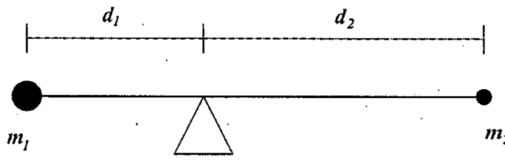


13 Mass, centre of mass and moments

By the end of this section, you should be able to answer the following questions:

- How can we use a double integral to find the mass of a two dimensional object if the density function is known?
- How do we use double integrals to locate the centre of mass of such an object?
- How do we calculate the moments of such an object about the coordinate axes?

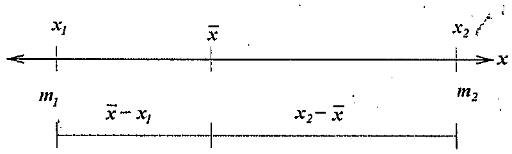
Ultimately we want to find a point P on which a thin plate of any given shape balances horizontally. Such a point is called the centre of mass of the plate.



Consider a rod of negligible mass balanced on a fulcrum. The rod has masses m_1 and m_2 at either end, which are a distance d_1 and d_2 respectively from the fulcrum. Because the rod is balanced, we have (thanks to Archimedes) the relationship

$$\boxed{m_1 d_1 = m_2 d_2.}$$

Now suppose the rod lies on the x-axis with m_1 at $x = x_1$, m_2 at $x = x_2$ and the centre of mass at $\overline{x}$.

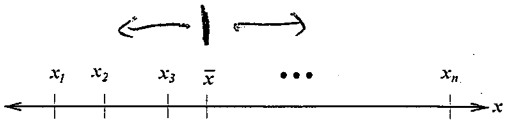


In this case we can write $d_1 = \overline{x} - x_1$ and $d_2 = x_2 - \overline{x}$, so Archimedes' relationship can be expressed

$$m_1(\overline{x} - x_1) = m_2(x_2 - \overline{x}) \;\Rightarrow\; \overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

The numbers $m_1 x_1$ and $m_2 x_2$ are called the *moments* of the masses m_1 and m_2 respectively.

$$(\bar{x}-x_1)m_1+(\bar{x}-x_2)m_2+\dots = \dots+(x_n-\bar{x})m_n$$

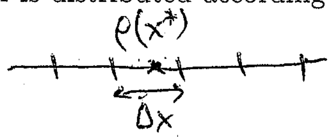


In general, a one dimensional system of n "particles" with masses $m_1, \dots, m_n$ located at $x = x_1, \dots, x_n$ has its centre of mass located at

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M}{m}$$

where $m = \sum m_i$ is the total mass of the system and the sum of the individual moments $M = \sum m_i x_i$ is called the moment of the system (with respect to the origin).

Now suppose the rod (which has length l) has mass which is distributed according to the (integrable) density function (mass/unit length)



1D !

$$\rho(x) = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x}.$$

Consider a small strip of width Δx containing the point x^*. The mass of this strip can be approximated by $\boxed{\rho(x^*)\Delta x.}$ Now cut the rod into n strips, and in the same way as above determine (approximately) the mass of each strip. To obtain an approximation for the total mass m of the rod, just add the masses of each n strips:

$$m \approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x_i.$$

To obtain a precise expression for the mass, we take the limit of this sum as $n \to \infty$. In other words,

$$m = \int_0^l \rho(x)dx.$$

We have a similar construction for the moment of the system. Consider the moment of each strip $\approx x_i^* \rho(x_i^*)\Delta x_i$. If we add these, we obtain an approximate expression for the moment of the system:

$$M \approx \sum_{i=1}^{n} x_i^* \rho(x_i^*)\Delta x_i.$$

Taking the limit as $n \to \infty$ we obtain an expression for the moment of the system about the origin:

$$M = \int_0^l x\rho(x)dx.$$

The centre of mass is located at $\overline{x} = M/m.$

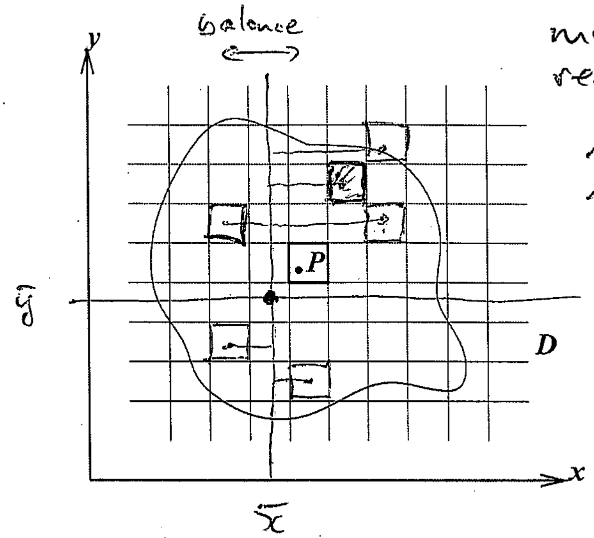
Now let's generalize this to two dimensions.

Suppose the lamina occupies a region D in the x-y plane and its density (in units of mass/unit area) is given by an integrable function $\rho(x, y)$. In other words,

2D

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A},$$

where Δm and ΔA are the mass and area of a small rectangle containing the point (x, y), and the limit is taken as the dimensions of $\Delta A \to 0$.



mass of one rectangle $\approx \rho(x^*, y^*)\Delta A$

Figure 26: The point $P = (x_i^*, y_j^*)$ in the rectangle R_{ij}.

To approximate the total mass of the lamina, we partition D into small rectangles (say R_{ij}) and choose a point (x_i^*, y_j^*) inside R_{ij}. The mass of the lamina inside R_{ij} is approximately $\rho(x_i^*, y_j^*)\Delta A_{ij}$, where ΔA_{ij} is the area of R_{ij}. Adding all such masses, we have the approximation

$$m \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_i^*, y_j^*)\Delta A_{ij}.$$

If we then take the limit as $m, n \to 0$, we obtain

$$m = \iint_D \rho(x, y)\, dA.$$

x-dir

$$\sum_{\text{(right)}} \underbrace{(x^* - \bar{x})}_{\text{dist.}_1} \underbrace{\rho(x^*, y^*)\Delta A}_{\text{mass}_1}$$

$$\approx \sum_{\text{(left)}} \underbrace{(\bar{x} - x^*)}_{\text{dist.}_2} \underbrace{\rho(x^*, y^*)\Delta A}_{\text{mass } 2}.$$

$$\Rightarrow \sum_{\text{(all)}} x^* \rho(x^*, y^*)\Delta A \approx \bar{x}\left(\sum_{\text{(all)}} \rho(x^*, y^*)\Delta A\right)$$

$$\Rightarrow \bar{x} \approx \frac{\sum_{\text{(all)}} x^* \rho(x^*, y^*)\Delta A}{\sum_{\text{(all)}} \rho(x^*, y^*)\Delta A}.$$

In a similar way, we can determine the moment of the lamina about the x-axis to be

$$M_x = \iint_D y\rho(x,y)dA$$

and the moment of the lamina about the y-axis to be

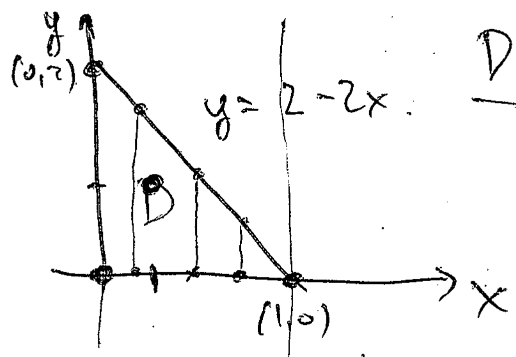
$$M_y = \iint_D x\rho(x,y)dA.$$

The centre of mass is located at coordinates $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}.$$

$$\boxed{\bar{x} = \frac{\iint_D x\rho \, dA}{\iint_D \rho \, dA} \qquad \bar{y} = \frac{\iint_D y\rho \, dA}{\iint_D \rho \, dA}.}$$

13.1 Example: find the centre of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$ and $(0,2)$ with constant density ρ_0.



$D = \{(x,y) \mid 0 \le x \le 1, \ 0 \le y \le 2-2x\}$.

want $(\bar{x}, \bar{y})$

$$\text{mass} = \iint_D \rho \, dA = \rho_0 \left(\iint_D dA\right) = \rho_0 \times (\text{area of } D)$$

$$= \rho_0 \times \tfrac{1}{2}(2 \times 1)$$

$$= \rho_0.$$

$$\iint_D x\rho_0 \, dA = \rho_0 \int_0^1 \left(\int_0^{2-2x} x \, dy\right) dx$$

$$= \rho_0 \int_0^1 \left[xy\right]_{y=0}^{y=2-2x} dx$$

$$= \rho_0 \int_0^1 x(2-2x)\,dx = \dots = \frac{\rho_0}{3}.$$

$$\iint_D y\rho_0\,dA = \rho_0 \int_0^1 \left(\int_0^{2-2x} y\,dy \right) dx$$

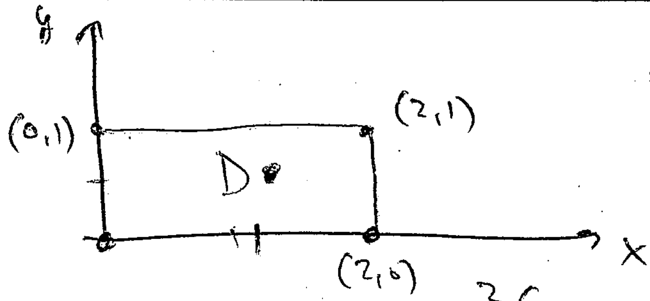
$$= \dots = \frac{2}{3}\rho_0$$

$$\Rightarrow \bar{x} = \frac{\iint_D x\rho\,dA}{\iint_D \rho\,dA} = \frac{\frac{1}{3}\rho_0}{\rho_0} = \frac{1}{3}$$

$$\bar{y} = \frac{\frac{2}{3}\rho_0}{\rho_0} = 2/3$$

13.2 Example: find the centre of mass of a rectangle with vertices $(0,0), (2,0)$, $(2,1)$ and $(0,1)$ with density $\rho(x,y) = 6x + 12y$.

$\rho(x,y) = 6x + 12y.$



$$\text{mass} = \iint_D \rho \, dA = \int_0^2 \left(\int_0^1 (6x+12y) \, dy \right) dx$$

$$= \int_0^2 \left[6xy + 6y^2 \right]_{y=0}^{y=1} dx.$$

$$= \ldots = 24. \; *$$

$$\iint_D x\rho \, dA = \int_0^2 \left(\int_0^1 (6x^2+12xy) \, dy \right) dx$$

$$= \ldots = 28$$

$$\iint_D y\rho \, dA = \int_0^2 \int_0^1 (6xy + 12y^2) \, dy \, dx$$

$$= \ldots = 14.$$

$$(\bar{x}, \bar{y}) = \left(\frac{28}{24}, \frac{14}{24} \right) = \left(\frac{7}{6}, \frac{7}{12} \right)$$