14 Introduction to Triple integrals

By the end of this section, you should be able to answer the following questions:

- How do you evaluate a triple integral?
- How do you use a triple integral to find the mass of a solid object with known density?
- How do you change the order of integration in a triple integral?

We can extend the definition of a double integral to a triple integral

$$\iiint_R f(x,y,z)dV, \overset{\text{def.}}{=} \lim_{\Delta V \to 0} \sum \underbrace{f(x^*, y^*, z^*)\Delta V}$$

where R is a region in $\mathbb{R}^3$ and dV is an element of volume. $(\Delta V = \Delta x \Delta y \Delta z)$

If R is a region in $\mathbb{R}^3$ specified by

(Fubini's theorem for $\iiint$, Stewart p1027)

$$\begin{array}{rcl} r(x,y) & \le z \le & s(x,y) \\ p(x) & \le y \le & q(x) \\ a & \le x \le & b \end{array} \qquad (11)$$

(e.g. mass density function $f(x,y,z) \geq 0$)

then

$$\iiint_R f(x,y,z)dV$$

$$= \int_a^b \left\{ \int_{p(x)}^{q(x)} \left[\int_{r(x,y)}^{s(x,y)} f(x,y,z)dz \right] dy \right\} dx.$$

In two dimensions, there are 2 possible orders of integration. In three dimensions, there are 6.

$dx\,dy\,dz$

$dx\,dz\,dy$

$\boxed{dy\,dx\,dz}$ *

$dy\,dz\,dx$

$dz\,dx\,dy$

$dz\,dy\,dx$

$$\left\{\begin{array}{l} c \le z \le d \\ f_1(z) \le x \le f_2(z) \\ g_1(x,z) \le y \le g_2(x,z) \end{array}\right.$$

14.1 **Find the mass of a rectangular block with dimensions $0 \leq x \leq L$, $0 \leq y \leq W$ and $0 \leq z \leq H$ of the density is** $\boxed{\rho = \rho_0 + \alpha xyz.}$

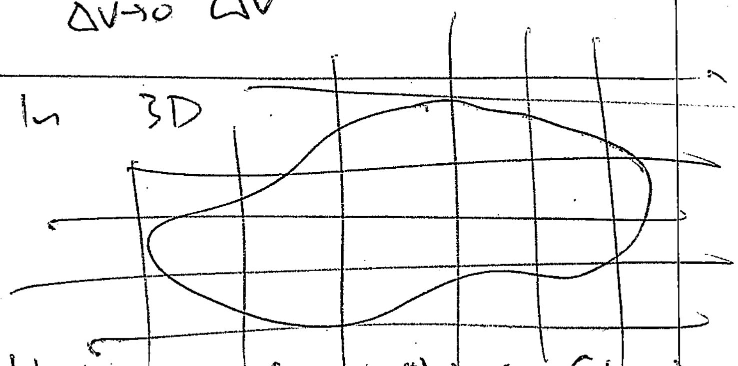
density function ρ gives
(3D)

mass per unit volume

$$\rho(x,y,z) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}.$$

In 3D

(x^*, y^*, z^*)



mass of one block $\approx \rho(x^*, y^*, z^*)\Delta V$ (cts ρ)

$\longrightarrow$ total mass $\approx \sum \rho(x^*, y^*, z^*)\Delta V$

$\lim_{\Delta V \to 0} \longrightarrow$ mass $= \iiint_V \rho(x,y,z)\, dV.$

$V = \{(x,y,z) \mid 0 \leq x \leq L,\ 0 \leq y \leq W,\ 0 \leq z \leq H.$

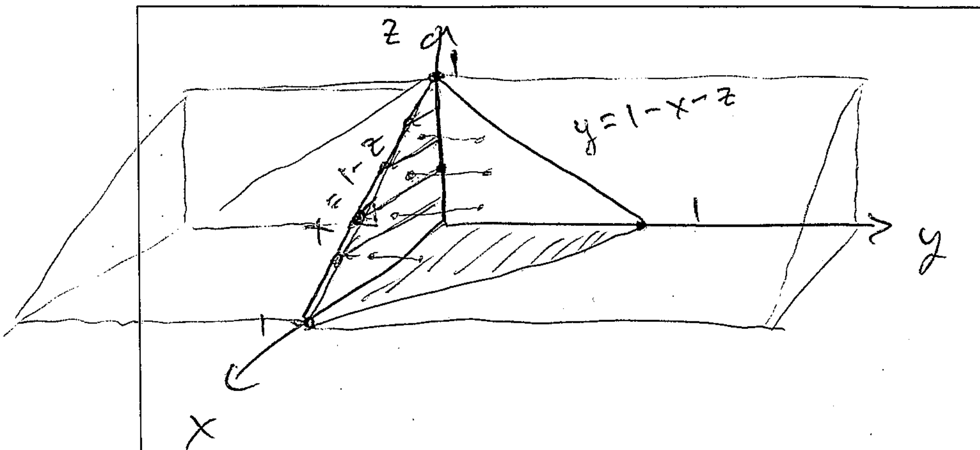
$$\text{mass} = \iiint_V \rho\, dV = \int_0^H \int_0^W \left(\int_0^L (\rho_0 + \alpha xyz)\, dx \right) dy\, dz$$

$$= \int_0^H \int_0^W \left[\rho_0 x + \tfrac{1}{2}\alpha x^2 yz \right]_{x=0}^{x=L} dy\, dz.$$

$$= \int_0^H \left(\int_0^W \left(\rho_0 L + \tfrac{1}{2}\alpha L^2 yz \right) dy \right) dz$$

$$= \ldots = \underbrace{\rho_0 LWH}_{\substack{\text{constant density} \\ \text{term = "density" × "volume".}}} + \frac{1}{8}\alpha L^2 W^2 H^2$$

14.2 Evaluate $\iiint_R z\, dV$ over the region R bounded by the surfaces $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.



$$R = \{(x,y,z) \mid 0 \le z \le 1,\ 0 \le x \le 1-z,\ 0 \le y \le 1-x-z\}.$$

$$\iiint_R z\, dV = \int_0^1 \int_0^{1-z} \left(\int_0^{1-x-z} z\, dy \right) dx\, dz$$

$$= \int_0^1 \int_0^{1-z} \Big[yz \Big]_{y=0}^{y=1-x-z} dx\, dz.$$

$$= \int_0^1 \left(\int_0^{1-z} \big((1-x-z)z - 0 \big)\, dx \right) dz.$$

$$= \cdots = \frac{1}{24}.$$

14.3 Changing the order of integration

Express the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\; dz\; dy\; dx$, in the orders $\underline{dz\; dx\; dy}$ and $dy\; dz\; dx$. Q33 p1035 Stewart.

$\underline{dz\, dx\, dy}$:

$$I = \int_0^1 \int_{\sqrt{x}}^1 \underbrace{\left(\int_0^{1-y} f(x,y,z)\, dz \right)}_{g(x,y)} dy\, dx$$

$$= \int_0^1 \int_{\sqrt{x}}^1 g(x,y)\; dy\; dx.$$

$$\ldots = \int_0^1 \int_0^{y^2} g(x,y)\; dx\; dy$$

$\underline{dy\, dz\, dx}$: $I = \int_0^1 \left(\int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy \right) dx$

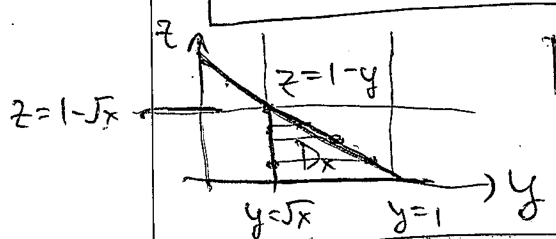
Set $h(x) = \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy.$

change to $dy\; dz$.

For $\underline{\text{any}}$ $x \in [0,1]$. we have

$D_x = \{(y,z) \mid \sqrt{x} \leq y \leq 1,\; 0 \leq z \leq 1-y\}.$

$\boxed{\text{Treat } x \text{ as a constant}}$



$D_x = \{(y,z) \mid 0 \leq z \leq 1-\sqrt{x},\; \sqrt{x} \leq y \leq 1-z\}.$

$$\Rightarrow \boxed{I = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dz\, dx}$$