15 Cylindrical coordinates

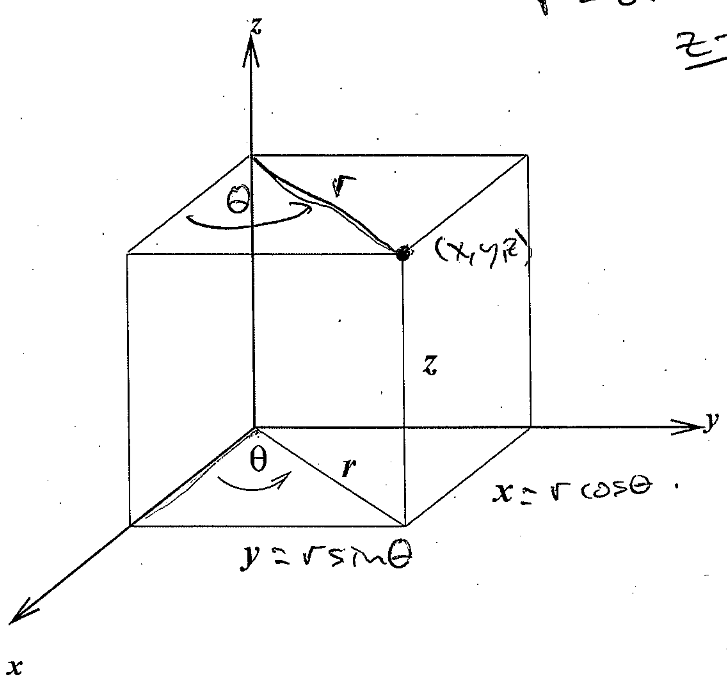
By the end of this section, you should be able to answer the following questions:

- What is the relationship between rectangular coorindates and cylindrical co-ordinates?
- How do you transform a triple integral in rectangular coordinates into one in terms of cylindrical coordinates?
- What is the Jacobian of the transformation?

Sometimes it is useful to use cylindrical coordinates in order to simplify the integral. This involves the transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{12}$$

r = distance to z-axis.



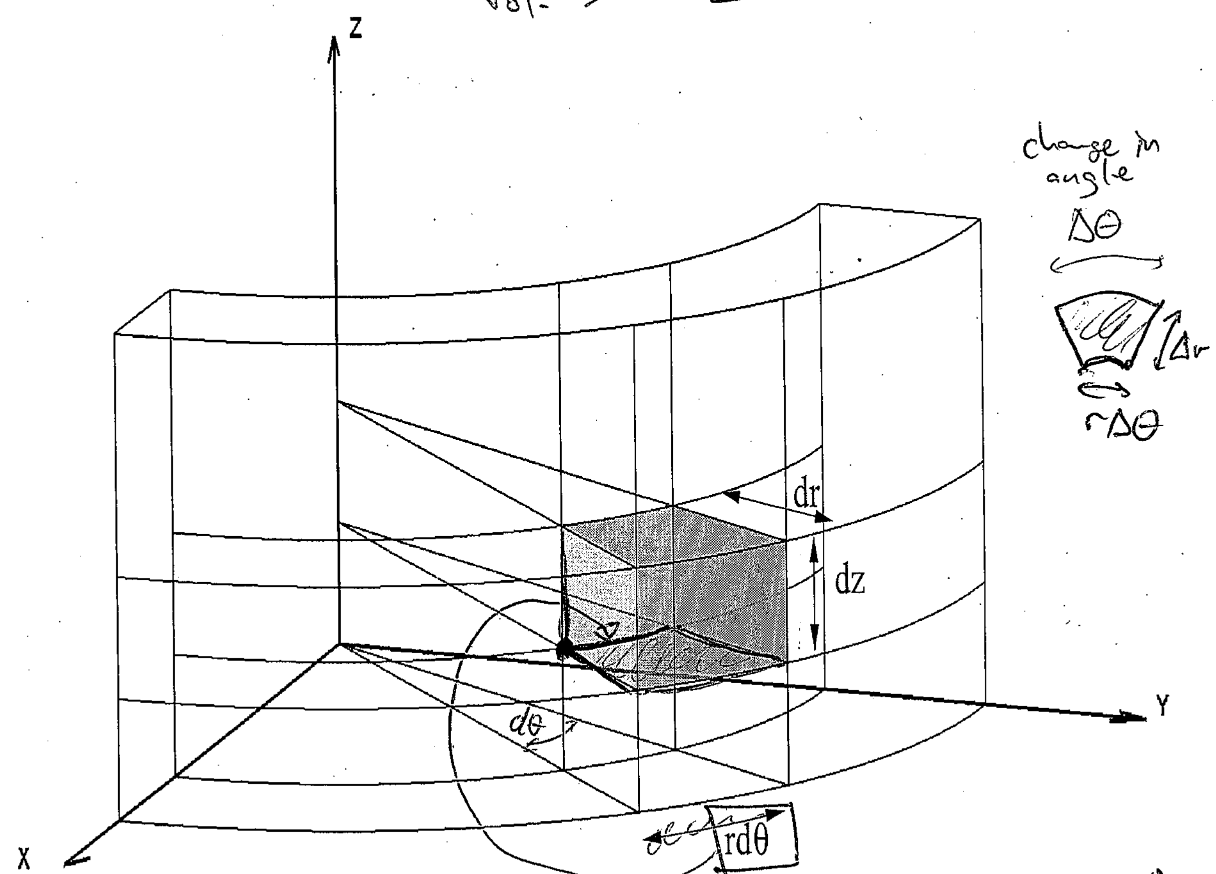
We now aim to calculate a small element of volume of a cylindrical shell. This will then show how in a triple integral we can transform from rectangular coordinates to cylindrical coordinates by substituting the transformation (12) and by making the change

$$dx\ dy\ dz \quad \longrightarrow \quad r\ dr\ d\theta\ dz.$$

Consider the following diagram.

area of base $\approx \Delta r \times r\Delta\theta$ (see polar coords)

vol. $\approx r\Delta\theta\,\Delta r\,\Delta z$.



(mass) $\approx \sum \rho(x^*, y^*, z^*)\Delta V \approx \sum \rho(r^*, \theta^*, z^*)\, r^* \Delta\theta\, \Delta r\, \Delta z$

The important result is that the triple integral in rectangular coordinates transforms as follows: $dx\ dy\ dz \longrightarrow r\ dr\ d\theta\ dz$

$$\iiint_R f(x,y,z)\ dx\ dy\ dz = \iiint_C f(r\cos\theta, r\sin\theta, z)\ \boxed{r}\ dr\ d\theta\ dz.$$

"Jacobian of transformation".

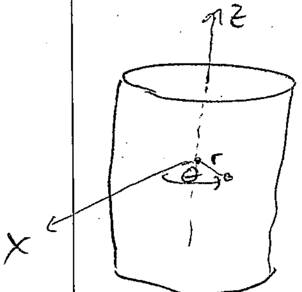
in 3D

$$\left.\begin{aligned} x &= x(u,v,w) \\ y &= y(u,v,w) \\ z &= z(u,v,w) \end{aligned}\right\} \text{Jac.} = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \right|$$

15.1 **A simple example: Find the volume of a cylinder of radius R and height H. (Ans. $\pi R^2 H$)**

Volume $V = \iiint_C dV \quad \left(= \lim_{\Delta V \to 0} \sum \Delta V \right)$

Cylinder $C = \{ (r, \theta, z) \mid 0 \le r \le R,\ 0 \le \theta \le 2\pi,\ 0 \le z \le H. \}$



$$\mathrm{Vol} = \int_0^H \int_0^R \int_0^{2\pi} r \, d\theta \, dr \, dz$$

cf. (prop. p59)

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^R r \, dr \right) \left(\int_0^H dz \right)$$

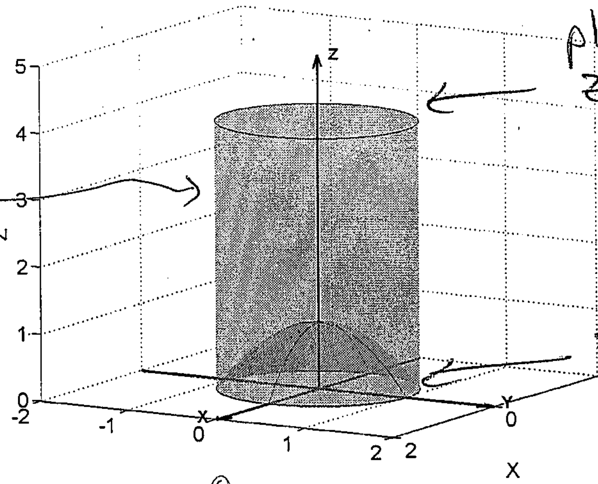
$$= 2\pi \times \frac{1}{2} R^2 \times H.$$

$$= \pi R^2 H.$$

15.2 Find the mass of the solid defined by the region contained within the cylinder $x^2+y^2=1$ below the plane $z=4$ and above the paraboloid $z=1-x^2-y^2$. The density at any given point in the region is proportional to the distance from the axis of the cylinder.

Use cylindrical coords. $\boxed{\rho = kr}$

cylinder $r=1$



plane $z=4$

$z=1-x^2-y^2$
$=1-r^2$

$x = r\cos\theta,\ y = r\sin\theta,\ z.\ ,\ x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$$\text{mass} = \iiint_V \rho \, dV$$

$$V = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 1-r^2 \le z \le 4\}.$$

$$\Rightarrow \iiint_V dV = \int_0^{2\pi}\left(\int_0^1 \int_{1-r^2}^4 kr \cdot r \, dz\, dr\right) d\theta$$

(prop p59) $$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 \int_{1-r^2}^4 kr^2 \, dz\, dr\right)$$

$$= 2\pi k \int_0^1 \left(\int_{1-r^2}^4 r^2 \, dz\right) dr$$

$$= 2\pi k \int_0^1 \left[r^2 z\right]_{z=1-r^2}^{z=4} dr$$

$$= 2\pi k \int_0^1 \left(4r^2 - r^2(1-r^2)\right) dr$$

$$= \ldots = \frac{12}{5}\pi k.$$