16 Spherical coordinates

By the end of this section, you should be able to answer the following questions:

- What is the relationship between rectangular coorindates and spherical coordinates?
- How do you transform a triple integral in rectangular coordinates into one in terms of spherical coordinates?
- What is the Jacobian of the transformation?

Sometimes it is useful to use spherical coordinates in order to simplify the integral. This involves the transformation

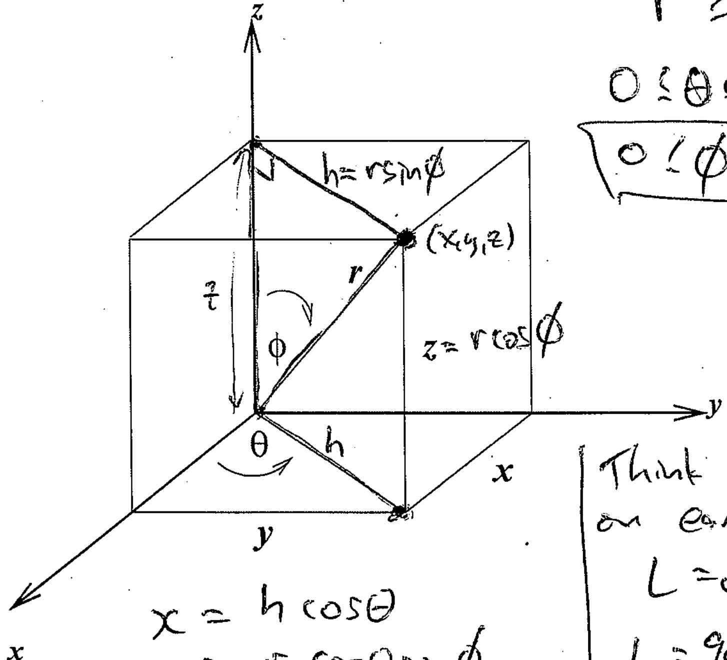
$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi. \tag{13}$$

In this case θ is longitude, ϕ is co-latitude, and r the distance from the origin.

$r \geq 0$

$0 \leq \theta < 2\pi$.

$\boxed{0 \leq \phi \leq \pi}$.



$x = h\cos\theta$
$= r\cos\theta\sin\phi$

$y = h\sin\theta$
$= r\sin\theta\sin\phi$.

Think of latitude L on earth's surface
$L = 0$ equator.
$L = 90°$ north pole
$L = -90$ south pole.
$L_{Bris.} \simeq -27°$

$\phi_{Bris.} \simeq 117° \simeq \frac{13\pi}{20}$.

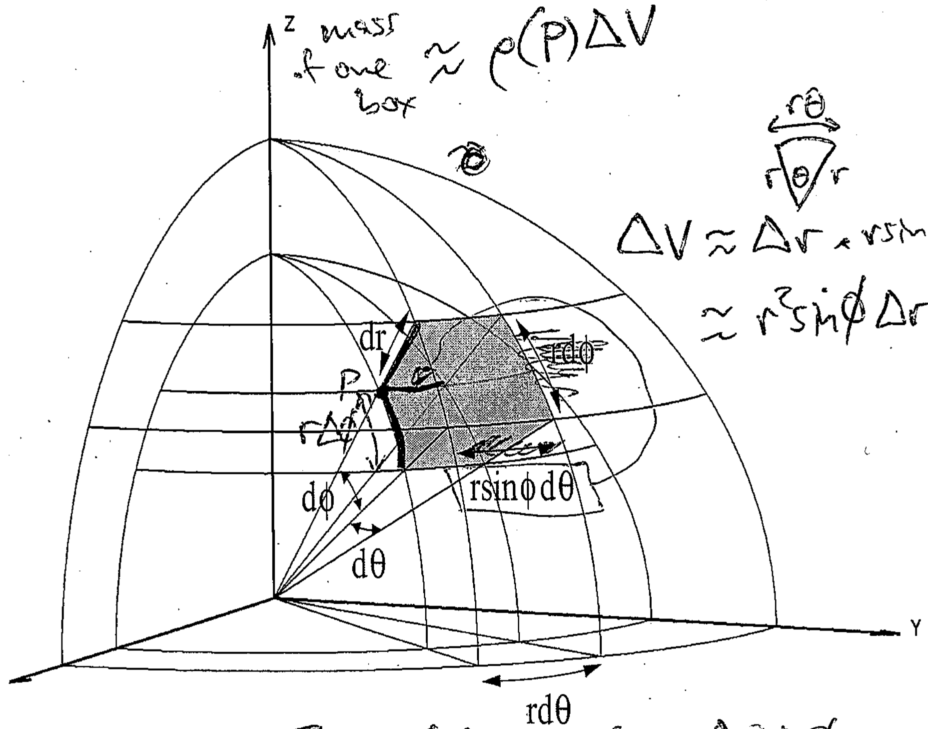
We now aim to calculate a small element of volume of a spherical shell. This will then show how in a triple integral we can transform from rectangular coordinates to spherical coordinates by substituting the transformation (13) and by making the change

$$\underline{dx\ dy\ dz} \longrightarrow \underline{r^2 \sin\phi\ dr\ d\theta\ d\phi}.$$

Consider the following diagram.

e.g. mass density function $\rho(x,y,z)$

mass of one box $\approx \rho(P)\Delta V$



$\Delta V \approx \Delta r \cdot r\sin\phi\Delta\theta \cdot r\Delta\phi$

$\approx r^2 \sin\phi\, \Delta r\, \Delta\theta\, \Delta\phi.$

$\Longrightarrow$ mass $\approx \sum_{(\text{boxes})} \rho(P)\, r^2 \sin\phi\, \Delta r\, \Delta\theta\, \Delta\phi.$

The important result is that the triple integral in rectangular coordinates transforms as follows:

$$\iiint_R f(x,y,z)\ dx\ dy\ dz$$
$$= \iiint_S f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)\boxed{r^2\sin\phi}\,dr\ d\theta\ d\phi.$$

"Jacobian".

16.1 A simple example: Find the volume of a sphere of radius R. (assume centred at $(0,0,0)=(x,y,z)$)

$$\text{Vol.} = \iiint_S dV \qquad \text{sphere } S.$$

$$S = \{(r,\theta,\phi) \mid 0 \le r \le R,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.\}.$$

$$\Longrightarrow \text{vol.} = \int_0^R \int_0^{2\pi} \int_0^{\pi} \underbrace{r^2 \sin\phi}_{\text{Jacobian}}\, d\phi\, d\theta\, dr$$

$$= \left(\int_0^R r^2\,dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\theta\right)$$

$$= \frac{1}{3}R^3 \times 2\pi \times -\cos\phi \Big|_0^{\pi}$$

$$= \frac{2}{3}\pi R^3 \times \underbrace{\left(-\cos\pi - (-\cos 0)\right)}_{=2}$$

$$= \frac{4}{3}\pi R^3.$$

16.2 Find the mass of a sphere of radius R whose density is given by $\rho(x, y, z) = e^{-(x^2+y^2+z^2)^{1/2}}$. (assume centred at $(x,y,z) = (0,0,0)$)

use spherical coords:

$$\begin{cases} x = r\cos\theta\sin\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases}$$

$\Rightarrow x^2+y^2+z^2 = r^2$

$\Rightarrow \rho(r,\theta,\phi) = e^{-r}$

$$\text{mass} = \iiint_S \rho \, dV = \int_0^R \int_0^{2\pi} \int_0^{\pi} e^{-r} r^2 \sin\phi \, d\phi \, d\theta \, dr$$

$$= \left(\int_0^R r^2 e^{-r} dr\right) \times \left(\int_0^{2\pi} d\theta\right) \times \left(\int_0^{\pi} \sin\phi \, d\phi\right)$$

Note: $\int_0^R r^2 e^{-r} dr = (\text{check}) = 2 - \dfrac{R^2+2R+2}{e^R}$

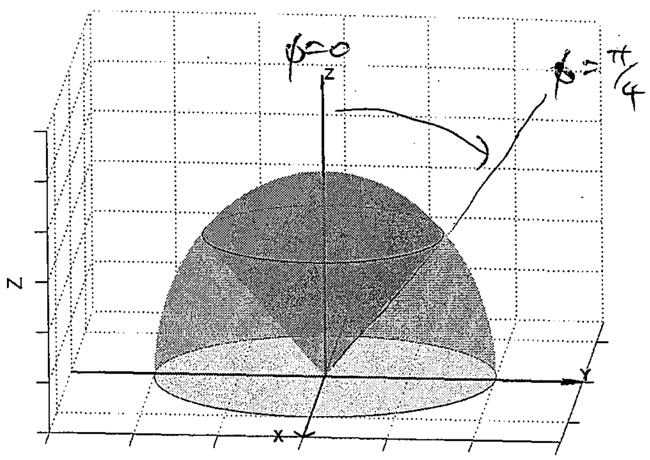
(use integration by parts twice)

$\Rightarrow \text{mass} = \left(2 - \dfrac{R^2+2R+2}{e^R}\right) \times 2\pi \times 2$

(note: take limit $R \to \infty$

$\Rightarrow \text{mass} = 8\pi$)

16.3 **Find the volume of the "ice cream cone" R between a sphere of radius a (centred at the origin) and the cone $z = \sqrt{x^2 + y^2}$.**



Use spherical coords.

Cone $z = \sqrt{x^2+y^2}$

$\Rightarrow r\cos\phi = \sqrt{r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\sin^2\phi}$

$= \sqrt{r^2\sin^2\phi\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}}$

$= r\sin\phi.$ (origin $r=0$)

$\Rightarrow \tan\phi = 1$ (for $0 \le \phi \le \frac{\pi}{2}$, $z \ge 0$)

$\Rightarrow \phi = \pi/4.$

$\Rightarrow R = \{(r,\theta,\phi) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4\}.$

$$\text{vol.} = \iiint_V dV$$

$$\Rightarrow \text{vol.} = \int_0^a \int_0^{\pi/4} \int_0^{2\pi} r^2 \sin\phi \, d\theta \, d\phi \, dr$$

$$= \left(\int_0^a r^2 dr\right)\left(\int_0^{\pi/4} \sin\phi \, d\phi\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \frac{1}{3}a^3 \times -\cos\phi \Big|_0^{\pi/4} \times 2\pi .$$

$$= \frac{2\pi}{3}a^3 \left(-\frac{1}{\sqrt{2}} + 1\right)$$