17 Moments of inertia (second moments)

By the end of this section, you should be able to answer the following questions:

- How do you locate the centre of mass of a solid object using a triple integral?
- How do you calculate the moments of inertia about the three coordinate axes?

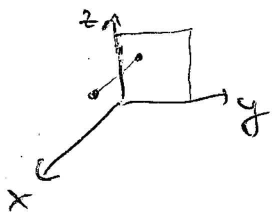
The moment of inertia of a particle of mass m about an axis (x, y, or z) is defined to be $\boxed{mr^2}$ where r is the distance from the particle to the axis.

It is sometimes referred to as rotational inertia and can be thought of as the rotational analogue of mass for linear motion. For example, linear kinetic energy can be expressed as $\frac{1}{2}mv^2$, and the rotational kinetic energy as $\frac{1}{2}I\omega^2$. Linear momentum is determined by the formula $p = mv$, while angular momentum is given by $L = I\omega$. In these examples, I is the moment of inertia and ω the angular velocity.

As we have seen from previous examples, the mass of a solid with density $\rho(x, y, z)$ occupying a region R in $\mathbb{R}^3$ is given by

$$m = \iiint\limits_R \rho(x, y, z)dV.$$

The moments about each of the three coordinate planes are



$$M_{yz} = \iiint\limits_R x\rho(x, y, z)dV, \quad M_{xz} = \iiint\limits_R y\rho(x, y, z)dV,$$

$$M_{xy} = \iiint\limits_R z\rho(x, y, z)dV$$

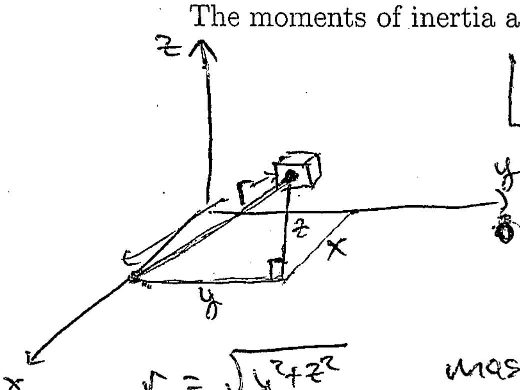
$$\bar{x} = \frac{\iiint\limits_R x\rho\, dV}{\iiint\limits_R \rho\, dV}$$

etc.

The centre of mass is then located at the point $(\overline{x}, \overline{y}, \overline{z})$ where

$$\overline{x} = \frac{M_{yz}}{m}, \quad \overline{y} = \frac{M_{xz}}{m}, \quad \overline{z} = \frac{M_{xy}}{m}.$$

The moments of inertia about each of the three coordinate axes work out to be



$$\boxed{I_x = \iiint\limits_R (y^2 + z^2)\rho(x, y, z)dV,}$$

$\lim \Delta V \to 0$

$$I_y = \iiint\limits_R (x^2 + z^2)\rho(x, y, z)dV,$$

$$I_z = \iiint\limits_R (x^2 + y^2)\rho(x, y, z)dV.$$

$r = \sqrt{y^2+z^2}$

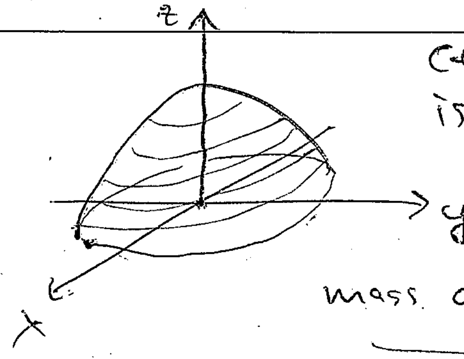
$r^2 = y^2+z^2$

mass of one box $\approx \rho(x^*, y^*, z^*)\Delta V$

$\Rightarrow$ moment of inertia $\approx (y^{*2}+z^{*2})\rho(x^*, y^*, z^*)\Delta V$

$\Rightarrow$ total moment of inertia (about x-axis) $\approx \sum\limits_{\text{(boxes)}} (y^2+z^2)\rho(x,y,z)\Delta V$

17.1 Example: locate the centre of mass of a solid hemisphere of radius a with density proportional to the distance from the centre of the base. Find its moment of inertia about the z-axis.



centre of sphere is located at $(x,y,z)=(0,0,0)$

Use spherical coords. (r,θ,ϕ)

mass density $\rho = kr$

Also $\bar{x}=\bar{y}=0$ by symmetry.

$\bar{z}$? mass $= \iiint_H \rho \, dV$

$$\bar{z} = \frac{\iiint_H z \rho \, dV}{\iiint_H \rho \, dV}$$

$H = \{(r,\theta,\phi) \mid 0 \le r \le a, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\}$.

$$\text{mass} = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} kr \cdot r^2 \sin\phi \, d\phi \, d\theta \, dr$$

$$= k\left(\int_0^a r^3 dr\right) \cdot \left(\int_0^{2\pi} d\theta\right) \cdot \left(\int_0^{\pi/2} \sin\phi \, d\phi\right)$$

$$= k \times \frac{1}{4}a^4 \times 2\pi \times \underbrace{\left[-\cos\phi\right]_0^{\pi/2}}_{=1}$$

$$= \frac{k\pi a^4}{2}$$

Also $\iiint_H z\rho\, dV = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} r\cos\phi \,.\, kr \,.\, r^2 \sin\phi \, d\phi \, d\theta \, dr$

$$= k\left(\int_0^a r^4 dr\right) \cdot \left(\int_0^{2\pi} d\theta\right) \times \left(\int_0^{\pi/2} \cos\phi \sin\phi \, d\phi\right)$$

$$= k \times \frac{1}{5}a^5 \times 2\pi \times \frac{1}{2}\sin^2\phi \Big|_0^{\pi/2}$$

$$= \frac{k\pi a^5}{5}.$$

$$\Rightarrow \bar{z} = \frac{\frac{k\pi a^5}{5}}{\frac{k\pi a^4}{2}} = \frac{2}{5}a.$$

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moment of inertia about z-axis

$$= \iiint_H (x^2+y^2)\rho\, dV \quad , \quad x = r\cos\theta \sin\phi$$
$$y = r\sin\theta \sin\phi.$$

$$\Rightarrow x^2+y^2 = r^2\sin^2\phi.$$

$$= \int_0^a \int_0^{2\pi} \int_0^{\pi/2} r^2\sin^2\phi \,.\, kr \,.\, r^2\sin\phi \, d\phi \, d\theta \, dr$$

$$= k\left(\int_0^a r^5 dr\right)\left(\int_0^{2\pi} d\theta\right)\underbrace{\left(\int_0^{\pi/2} \sin^3\phi \, d\phi\right)}_{\substack{\sin^3\phi = (1-\cos^2\phi)\sin\phi \\ \& \text{ set } u = -\cos\phi \ldots}}$$

$$= k \times \frac{1}{6}a^6 \times 2\pi \times \int_{-1}^{0} (1-u^2)\, du$$

$$= \frac{2\pi k a^6}{9}.$$