Stewart CH 17:

18 Conservative vector fields

By the end of this section, you should be able to answer the following questions:

- What is meant by a conservative vector field and a corresponding potential function? $\underset{\sim}{F} = \nabla f$
- Given a potential function, how do you determine the corresponding conservative vector field? Calculate ∇f.
- Given a conservative vector field, how do you determine a corresponding potential function?

18.1 Vector fields

In what follows, the notation is always

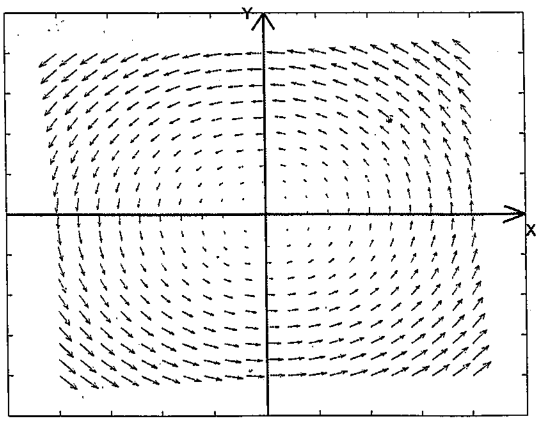
$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} \text{ or } \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

A vector field in the x-y plane is a vector function of 2 variables

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{F}(x,y) &= (F_1(x,y), F_2(x,y)) \\ &= F_1(x,y)\boldsymbol{i} + F_2(x,y)\boldsymbol{j}. \end{aligned}$$

That is, associated to a point (x,y) is the vector $\boldsymbol{F}(\boldsymbol{r})$.

18.1.1 Example: $\boldsymbol{F}(\boldsymbol{r}) = (-y, x) = -y\boldsymbol{i} + x\boldsymbol{j}$.



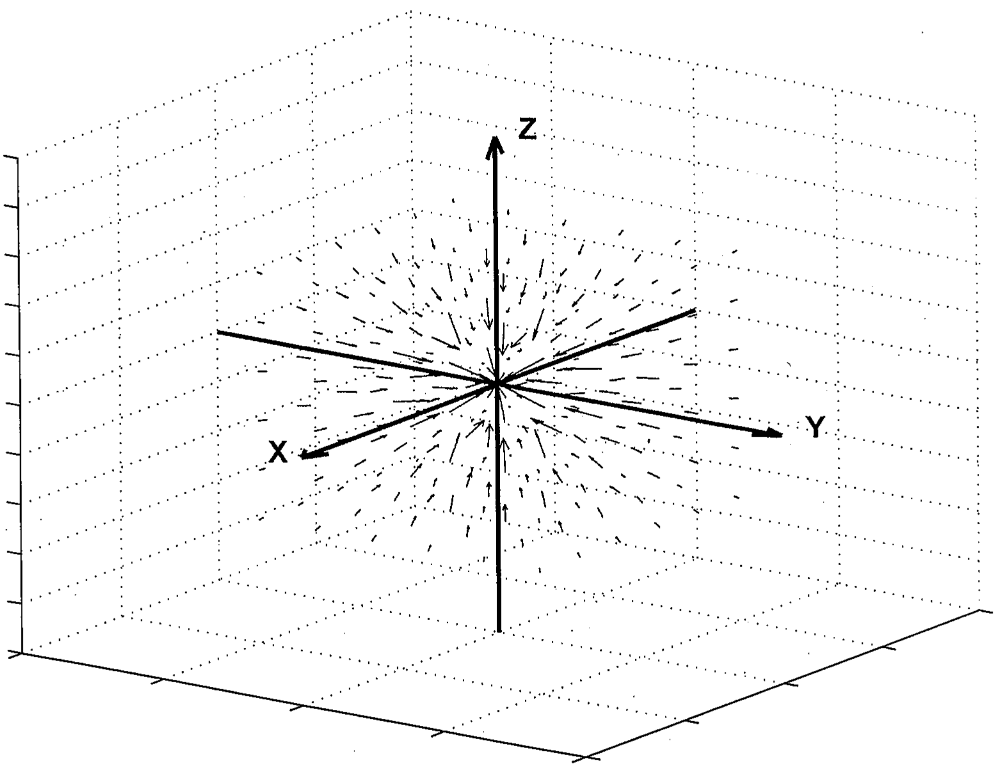
Similarly a vector field in 3-D is a vector function of 3 variables

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{r}) &= \boldsymbol{F}(x,y,z) \\
&= (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)) \\
&= F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}
\end{aligned}$$

18.1.2 Example: Newtonian gravitational field

$|\underline{r}| = \sqrt{x^2+y^2+z^2}$

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{r}) &= -\frac{mMG}{|\boldsymbol{r}|^3}\boldsymbol{r} = \boldsymbol{F}(x,y,z) \\
&= \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{i} + \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{j} \\
&\quad + \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}}\boldsymbol{k}
\end{aligned}$$



18.2 Gradient of a scalar field, conservative vector fields

Recall for a differentiable scalar function $f(x, y)$ in two dimensions, we define

$$\text{grad} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j. \quad = \underset{\sim}{\nabla} f$$

For a differentiable scalar function $f(x, y, z)$ in three dimensions, we define

$$\text{grad} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$$

Alternatively we define the differential operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

so grad$f = \nabla f$.

- directional derivatives $D_{\underset{\sim}{u}}(f(x,y)) = (\underset{\sim}{\nabla} f) \cdot \underset{\sim}{u}$
- find local max, min, saddle points.

18.2.1 Example: find the gradient of $f(x, y, z) = x^2y^3z^4$..

$$\underset{\sim}{\nabla} f = \underset{\sim}{i}\frac{\partial}{\partial x}(x^2y^3z^4) + \underset{\sim}{j}\frac{\partial}{\partial y}(x^2y^3z^4) + \underset{\sim}{k}\frac{\partial}{\partial z}(x^2y^3z^4)$$

$$= 2xy^3z^4\underset{\sim}{i} + 3x^2y^2z^4\underset{\sim}{j} + 4x^2y^3z^3\underset{\sim}{k}.$$

e.g. at $(1,1,1)$

$$\underset{\sim}{\nabla} f(1,1,1) = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}.$$

Direction of maximum increase of f.

$|\underset{\sim}{\nabla} f|_{(1,1,1)} = \sqrt{29} \longrightarrow$ maximum rate of increase.

Note ∇f is a vector. It's length and direction are independent of the choice of coordinates. ∇f (evaluated at a given point P) is in the direction of maximum increase of f at P. $\longrightarrow$ perpendicular to level curves/surfaces (2D/3D)

You may see the scalar function f referred to as a scalar field. If a vector field v and a scalar field f are related by $\boxed{v = \nabla f,}$ we call f a potential function and v a *conservative vector field.*

$\rightarrow$ Challenge: express $\underset{\sim}{\nabla} f$ (in 2D) in terms of polar coords. (use $\underset{\sim}{\hat{r}}, \underset{\sim}{\hat{\theta}}$)

18.2.2 Verify that the Newtonian gravitational field is conservative with potential function $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$. $= mMG(x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial f}{\partial x} = -\frac{1}{2} mMG \cdot 2x\,(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2} mMG \cdot 2y\,(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial f}{\partial z} = -\frac{1}{2} mMG \cdot 2z\,(x^2+y^2+z^2)^{-3/2}$$

$$\Rightarrow \underline{\nabla} f = \frac{-mMG}{(x^2+y^2+z^2)^{3/2}}\left(x\underline{i} + y\underline{j} + z\underline{k}\right)$$

$$= \frac{-mMG}{|\underline{r}|^3}\,\underline{r}$$

Given a conservative vector field, how can we determine a corresponding potential function? The next example outlines this procedure.

18.2.3 **The vector field $\boldsymbol{F}(x,y) = (3+2xy)\boldsymbol{i} + (x^2-3y^2)\boldsymbol{j}$ is conservative. Find a corresponding potential function.**

$\Rightarrow \exists f$ such that $\underline{F} = \underline{\nabla} f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j}$

What is f?

$\Rightarrow \frac{\partial f}{\partial x} = 3 + 2xy$ ($\underline{i}$ comp's)

$\Rightarrow f(x,y) = 3x + x^2 y + g(y)$ (*)

$\Rightarrow \frac{\partial f}{\partial y} = 0 + x^2 + g'(y)$

$\stackrel{\text{set}}{=} x^2 - 3y^2$ ($\underline{j}$ comp of $\underline{F}$)

$\Rightarrow g'(y) = -3y^2$

$\Rightarrow g(y) = -y^3 + c$

(*) $\Rightarrow \boxed{f(x,y) = 3x + x^2 y - y^3 + c}$

Can we still determine a potential function when the conservative vector field is in three dimensions?

18.2.4 The vector field $\boldsymbol{F}(x, y, z) = y^2\boldsymbol{i} + (2xy + e^{3z})\boldsymbol{j} + 3ye^{3z}\boldsymbol{k}$ is conservative. Find a corresponding potential function.

$\Rightarrow \exists f$ s.t. $\underline{F} = \underline{\nabla} f$

$$\underline{\nabla} f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}.$$

$$\Rightarrow \frac{\partial f}{\partial x} = y^2 \quad (\underline{i} \text{ comp's})$$

$$\Rightarrow f = xy^2 + g(y,z) \quad (*)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2xy + \frac{\partial g}{\partial y} \stackrel{\text{set}}{=} 2xy + e^{3z} \quad (\underline{j} \text{ comp of } \underline{F})$$

$$\Rightarrow \frac{\partial g}{\partial y} = e^{3z} \Rightarrow g = ye^{3z} + h(z)$$

$$(*) \Rightarrow f = xy^2 + ye^{3z} + h(z) \quad (**)$$

$$\Rightarrow \frac{\partial f}{\partial z} = 0 + 3ye^{3z} + h'(z) \stackrel{\text{set}}{=} 3ye^{3z} \quad (\underline{k} \text{ comp. of } \underline{F})$$

$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C.$$

$$(**) \Rightarrow \boxed{f(x,y,z) = xy^2 + ye^{3z} + C}$$

Is there a way of determining whether or not a given vector field is conservative? To answer this question, we need to go back to the study of line integrals.