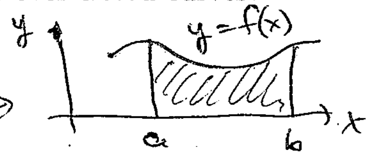
19 The fundamental theorem for line integrals, path independence

By the end of this section, you should be able to answer the following questions:

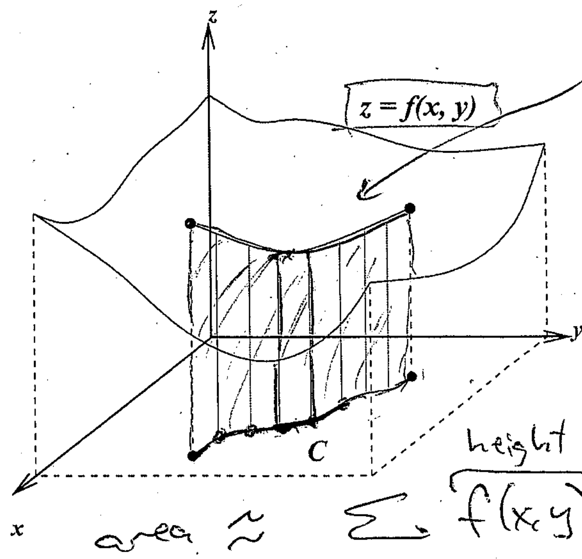
- How do you evaluate line integrals?
- What is the fundamental theorem for line integrals and its consequences?
- What is a path independent line integral and what are its connections with conservative vector fields and line integrals over closed curves?

19.1 Line integrals in the plane



Recall the definite integral $\boxed{\int_a^b f(x)\, dx}$ gives the net area above the x-axis and below its image $y = f(x)$. We can generalise this.

Consider the following problem: How do we calculate the area of the region between the curve C in the x-y plane and its image on the surface $z = f(x, y)$?



curve lies on surface.

area $\approx \sum \underbrace{f(x,y)}_{\text{height}} \underbrace{\Delta s}_{\text{width}}$

$\lim_{\Delta s \to 0} \sum f(x,y)\Delta s$ = area $\overset{\text{def.}}{=} \int_C f(x,y)\, ds$

If the curve C can be parametrised by $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$ for $a \le t \le b$, then the area is given by the formula

$$\text{area} = \underbrace{\int_C f(x, y)\, dS}_{\text{"line integral"}} = \int_a^b f(x(t), y(t))|\boldsymbol{r}'(t)|\, dt,$$

where dS is the infinitesimal element of arclength of C.

"line integral".

19.2 Line integrals of vector fields

smooth = cts, cts derivatives.

We can also consider integrating a vector field over a curve in the plane.

Let C be a piecewise continuous smooth curve in the x-y plane joining points A and B. Let $\boldsymbol{F}(x,y) = F_1(x,y)\boldsymbol{i} + F_2(x,y)\boldsymbol{j}$ be a vector field. A line integral is given by

convenient notation only

$$\begin{aligned} \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C (F_1(x,y)dx + F_2(x,y)dy) \\ &= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt \end{aligned}$$

eval.

Same quantity expressed in four ways.

where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$, $\boxed{d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j}}$ and x, y are parameterised by $t \in [a,b]$.

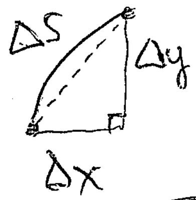
Note that we can also write the line integral as $\boxed{\int_C \boldsymbol{F}(x,y) \cdot \boldsymbol{T}(x,y)\, dS}$ where $\boldsymbol{T}$ is a unit tangent vector to the curve C at the point (x,y) on C. definition.

In the case $\boldsymbol{F}$ is a field of force, you should already be able to determine the work done by $\boldsymbol{F}$ in moving a particle along a curve C. Namely, you should already know that

In 1D constant force $W = Fd$.

$$\begin{aligned} \text{work} &= \int_a^b \left[\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} \right] |\boldsymbol{r}'(t)| dt \\ &= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt. \end{aligned}$$

19.3 Evaluating line integrals



In general, to evaluate a line integral

$$\int_C f(x,y)\, dS,$$

$$\Delta S \approx \underbrace{\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}}_{\lim_{\Delta t \to 0} \to |\boldsymbol{r}'(t)|} \Delta t$$

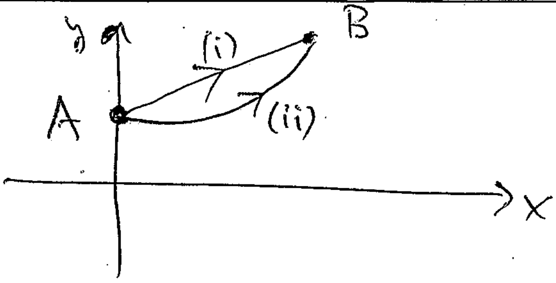
which includes line integrals of the form

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|}\, dS,$$

we start by parametrising C with $\boldsymbol{r}(t)$ and in the integral $\boxed{\text{replace } dS \text{ by } |\boldsymbol{r}'(t)|\, dt.}$ Then evaluate the integral as a definite integral in t. The bounds of integration for t are those values corresponding to the endpoints of C.

$\underline{F} = (x^2-y)\underline{i} + (y^2+x)\underline{j}$

19.3.1 Example: let $A = (0,1)$, $B = (1,2)$. Evaluate $\int_C ((x^2-y)dx + (y^2+x)dy)$ along the curve C given by: (i) the straight line from A to B; (ii) the parabola $y = x^2+1$ from A to B.



(i) Line from A ~~A~~ to B $\boxed{y = x+1}$

parameterise curve

$$\underline{r}(t) = \underbrace{t}_{x(t)}\,\underline{i} + \underbrace{(t+1)}_{y(t)}\,\underline{j}, \quad \boxed{0 \le t \le 1}$$

$$\int_0^1 \left((x^2-y)\frac{dx}{dt} + (y^2+x)\frac{dy}{dt}\right)dt \quad *$$

$$= \int_0^1 \left[\left(t^2-(t+1)\right).1 + \left((t+1)^2+t\right).1\right]dt$$

$$= \dots = 5/3.$$

(ii) along $\boxed{y = x^2+1}$

$$\underline{r}(t) = t\,\underline{i} + (t^2+1)\,\underline{j}, \quad 0 \le t \le 1.$$

$$x = t, \quad \frac{dx}{dt} = 1, \quad y = t^2+1, \quad \frac{dy}{dt} = 2t.$$

$$\int_C = \int_0^1 \left[\left(t^2 - (t^2+1)\right)\cdot 1 + \left((t^2+1)^2 + t\right)2t\right] dt$$

$$= \ldots = 2.$$

Note the line integrals in the previous example were *path dependent.* In other words, they have different values for different paths.

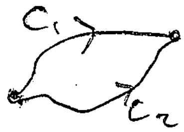
We will now investigate path independent line integrals.

19.4 Line integrals of conservative vector fields, path independence.

$\underline{F} = \nabla f$

If $\boldsymbol{F}$ is a continuous vector field with domain D, we say the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is *path independent* if

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$$



for **any** two paths C_1 and C_2 in D that have the same end points.

19.4.1 The fundamental theorem for line integrals

If C is a smooth curve determined by $\boldsymbol{r}(t)$ for $t \in [a, b]$ and $f(x, y)$ is differentiable with ∇f being continuous on C, then

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a)).$$

Proof: $\quad d\underline{r} = dx\,\underline{i} + dy\,\underline{j}$

$$\underline{\nabla f} = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} \qquad f(x(t), y(t))$$

$$\Rightarrow \int_C (\underline{\nabla f}) \cdot d\underline{r} = \int_a^b \Big(\underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}_{\text{(chain rule)}} \Big) dt$$

$$= \int_a^b \frac{df}{dt}\, dt$$

$$= \int_{f(\underline{r}(a))}^{f(\underline{r}(b))} df = f(\underline{r}(b)) - f(\underline{r}(a))$$

One consequence is that for conservative vector fields ∇f, we have

$$\int_{C_1} \nabla f \cdot d\boldsymbol{r} = \int_{C_2} \nabla f \cdot d\boldsymbol{r}.$$

That is, the line integral of a conservative vector field is path independent.

It turns out, the converse is also true. Suppose $\boldsymbol{F}$ is continuous on an open, connected region D. If $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is path independent in D, then $\boldsymbol{F}$ is conservative.

Proof:

Stewart p1084-1085.
(Theorem 4).

Open region: every point in the region is the centre of some disc lying entirely in the region (ie. an open region doesn't include the boundary points).

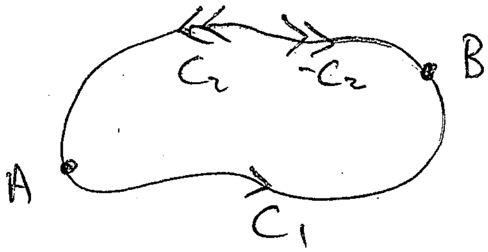
Connected region: Any two points in D can be joined by a path lying entirely in D.

Another interesting result is that if $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ is path independent in some region D, then $\displaystyle\oint_{C'} \boldsymbol{F}\cdot d\boldsymbol{r} = 0$ for *every* closed path C' in D. Here the symbol "$\oint$" indicates the integral is over a *closed* curve.

Proof:

Assume path independence.

Let C' be a closed curve.



$$\left.\begin{array}{l} C_1 : A \to B \\ C_2 : B \to A \\ -C_2 : A \to B \end{array}\right\} \text{ as indicated.}$$

$*\left(\text{Note: } \displaystyle\int_{C_2} \underline{F}\cdot d\underline{r} = -\int_{-C_2} \underline{F}\cdot d\underline{r}\right)*$

~~Let C' be~~ $C' = C_1$ then C_2

$$\Rightarrow \oint_{C'} \underline{F}\cdot d\underline{r} = \int_{C_1} \underline{F}\cdot d\underline{r} + \int_{C_2} \underline{F}\cdot d\underline{r} \quad *$$

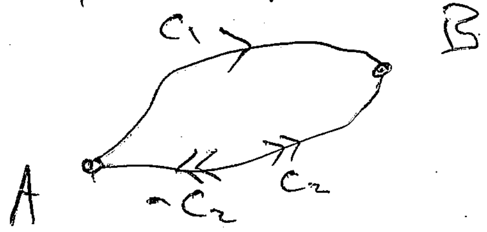
$$= \int_{C_1} \underline{F}\cdot d\underline{r} - \int_{-C_2} \underline{F}\cdot d\underline{r}$$

$$= 0 \quad \text{(by path independence)}$$

Perhaps it is not surprising that the converse is also true. That is, if $\oint_{C'} \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for *every* closed path C' in some region D, then $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is path independent in D.

Proof:

Take any two paths from A to B.



define $C = C_1$ then $-C_2$

$$0 = \oint_C \underline{F} \cdot d\underline{r} \quad \text{(by assumption)}$$

$$= \int_{C_1} \underline{F} \cdot d\underline{r} + \int_{-C_2} \underline{F} \cdot d\underline{r}$$

$$= \int_{C_1} \underline{F} \cdot d\underline{r} - \int_{C_2} \underline{F} \cdot d\underline{r}.$$

Since C_1, C_2 arbitrary

$\Rightarrow$ path independence.

We are looking at these results carefully because we ultimately want a simple way of checking whether or not a vector field is conservative. We are not quite there yet, but in the next section, we will arrive at a surprisingly simple test for a conservative vector field.

Note also that more details of these proofs (with slightly more mathematical rigour) can be found in Stewart, pages ~~1110 – 1113.~~ 1083 – 1085