"Advanced Engineering Mathematics" by E. Kreyszig.
"Elementary differential equations & boundary value problems" by Boyce & Di Prima.

2 Exact first order ODEs

By the end of this section, you should be able to answer the following questions about first order ODEs:

- How do you identify an exact ODE?
- How do you solve an exact ODE?
- ~~Under what conditions is a solution to an IVP problem unique?~~

2.1 Definition

First recall that if $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t, then z is a differentiable function of t whose derivative is given by the chain rule:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Now suppose the equation

$$f(x, y) = C$$

defines y implicitly as a function of x (here C is a constant). Then $y = y(x)$ can be shown to satisfy a first order ODE obtained by using the chain rule above. In this case, $z = f(x, y(x)) = C$, so

$$\begin{aligned} (0 = \ ) \quad \frac{dz}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} \\ \Rightarrow \boxed{f_x + f_y y' = 0.} & \qquad (1) \end{aligned}$$

A first order ODE of the form

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0 \qquad (2)$$

is called exact if there is a function $f(x, y)$ (compare (2) with (1) above) such that

$$f_x(x, y) = P(x, y) \text{ and } f_y(x, y) = Q(x, y).$$

The solution is then given implicitly by the equation

$$f(x, y) = C.$$

The constant C can usually be determined by some kind of "initial condition".

Given an equation of the form (2), how do we determine whether or not it is exact? There is a simple test.

Consider $\frac{\partial P}{\partial y} = \frac{\partial f}{\partial y \partial x}$ ← equal if cts

$\frac{\partial Q}{\partial x} = \frac{\partial f}{\partial x \partial y}$ ↙ (Clairaut's thm p921 of Stewart)

2.2 Test for exactness

Let P, Q, $\frac{\partial P}{\partial y}$, and $\frac{\partial Q}{\partial x}$ be continuous over some region of interest. Then

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

is an exact ODE iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

everywhere in the region.

The problem of actually determining $f(x,y)$ is still outstanding. Consider the following example.

2.3 Example: $\overbrace{2x + e^y}^{P} + \overbrace{xe^y}^{Q} y' = 0$ exact ?

$\frac{\partial P}{\partial y} = e^y$, $\frac{\partial Q}{\partial x} = e^y$ $\Rightarrow$ exact.

$\Rightarrow \exists f(x,y)$ s.t. ....

$P = \frac{\partial f}{\partial x} = 2x + e^y$.

integrate partially w.r.t. x.

$\Rightarrow f(x,y) = x^2 + xe^y + \underbrace{g(y)}$

↑similar to a constant of integration.

$\Rightarrow \frac{\partial f}{\partial y} = 0 + xe^y + g'(y)$

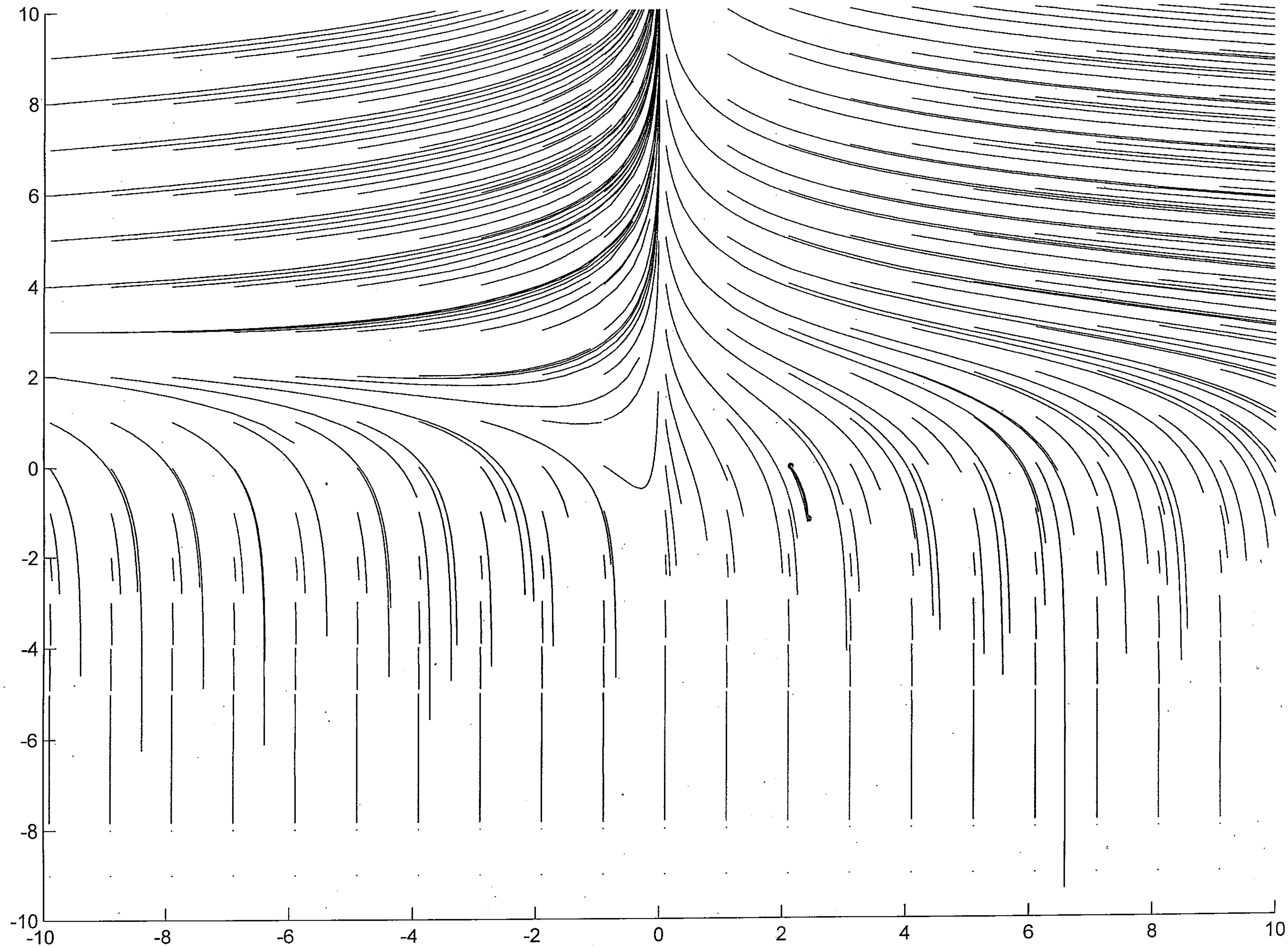
set $= Q = xe^y$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = \text{const.}$

$\Rightarrow f(x,y) = x^2 + xe^y + C$.

Implicit solution is $x^2 + xe^y = K$.

$\Rightarrow y = \ln\left(\frac{K - x^2}{x}\right)$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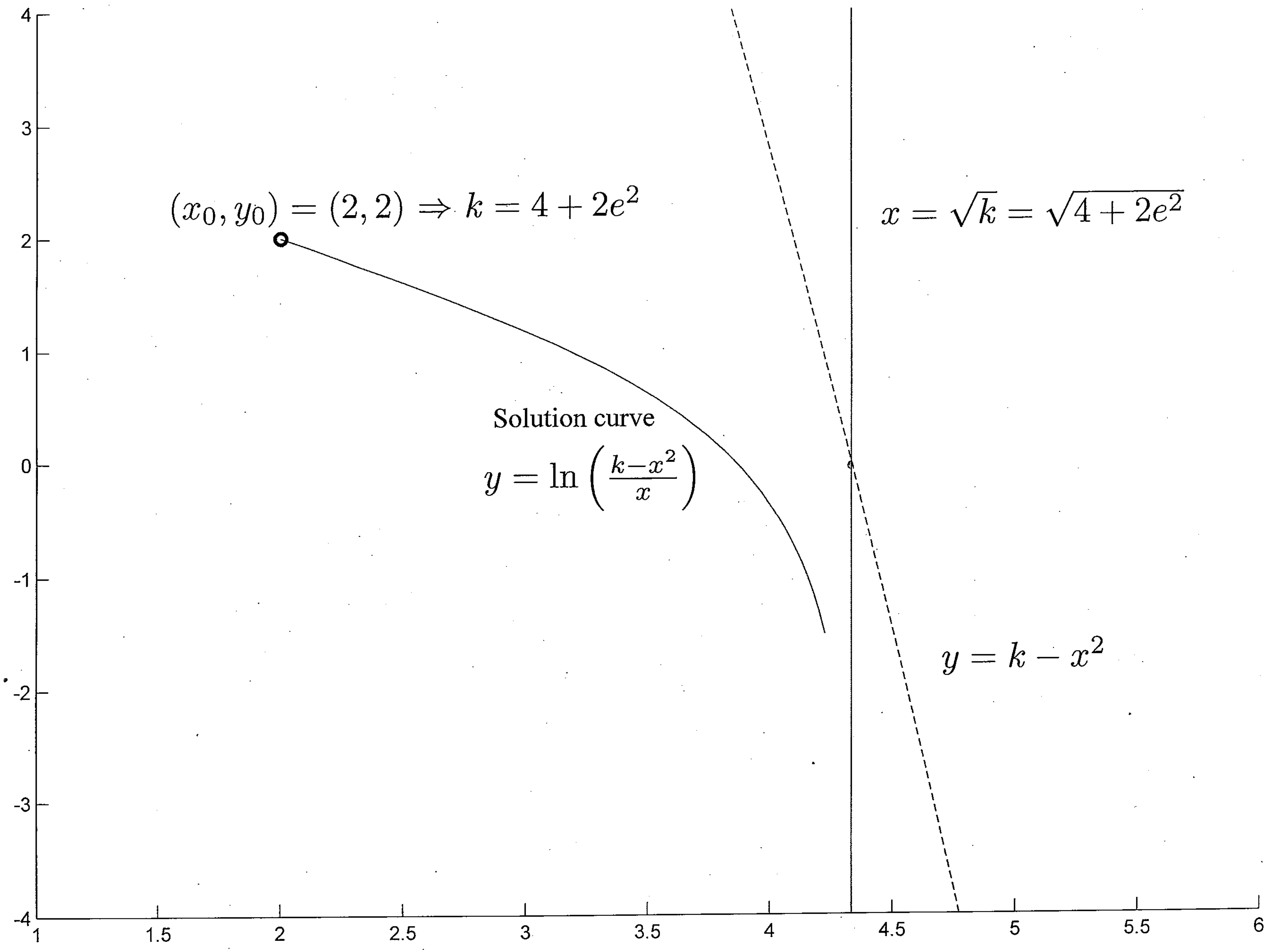

$(x_0, y_0) = (2, 2) \Rightarrow k = 4 + 2e^2$
$x = \sqrt{k} = \sqrt{4 + 2e^2}$
Solution curve
$y = \ln\left(\frac{k-x^2}{x}\right)$
$y = k - x^2$
4
3
2
1
0
-1
-2
-3
-4
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6

2.4 Almost exact ODEs and integrating factors

Let's say that we have an equation

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

such that

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}.$$

The test we have just seen tells us that the ODE is not exact. Are we still able to do anything with it? Here we consider using an "integrating factor", which is different to the one introduced to solve linear ODEs.

The idea is to multiply the ODE by a function $h(x,y)$ and then see if it is possible to choose $h(x,y)$ such that the resulting equation

$$h(x,y)P(x,y) + h(x,y)Q(x,y)\frac{dy}{dx} = 0$$

is exact. We know from the test that this new equation is exact if and only if

$$\frac{\partial}{\partial y}(hP) = \frac{\partial}{\partial x}(hQ).$$

Let's see if we can find such a function:

test $\Rightarrow$ $h_y P + h P_y = h_x Q + h Q_x$

$\Rightarrow$ $h_y P - h_x Q + h(P_y - Q_x) = 0$

(First order PDE in h !)

In general, the equation for $h(x,y)$ is usually just as difficult to solve as the original ODE. In some cases, however, we may be able to find an integrating factor which is a function of only one of the variables x or y. Let's try $\boxed{h \equiv h(x):}$

$$\Rightarrow \frac{dh}{dx} = h\left(\underbrace{\frac{P_y - Q_x}{Q}}\right)$$

If $\frac{P_y - Q_x}{Q}$ is a function of x only

this is separable & can be solved.

2.5 Example: $\overbrace{(3xy+y^2)}^{P}+\overbrace{(x^2+xy)}^{Q}\frac{dy}{dx}=0$

$P_y = 3x+2y$, $Q_x = 2x+y$

$\Rightarrow$ not exact. Try $h \equiv h(x)$

$\Rightarrow h(3xy+y^2) + h(x^2+xy)\frac{dy}{dx} = 0$

is exact if

$h(3x+2y) = h'(x^2+xy) + h(2x+y)$

$\Rightarrow h'\,x(x+y) = h(x+y)$

$\Rightarrow h' = \frac{h}{x}$.

$\Rightarrow h = ax$, any a, choose $a=1$

$\Rightarrow$ multiply original ODE by x.

$\Rightarrow \underbrace{3x^2y+xy^2}_{\bar{P}} + \underbrace{(x^3+x^2y)}_{\bar{Q}}\frac{dy}{dx} = 0$ --- $(*)$

$\bar{P}_y = 3x^2+2xy$, $\bar{Q}_x = 3x^2+2xy$.

$\therefore$ equ $(*)$ is exact.

Check solution $\boxed{x^3y+\frac{1}{2}x^2y^2=c}$

Note: This does not always work! (setting $h \equiv h(x)$)

This example is a special case.

EXPOSURE TO CONCEPT