20 Green's theorem and a test for conservative fields

By the end of this section, you should be able to answer the following questions:

- What is Green's theorem and under what conditions can it be applied?
- How do you apply Green's theorem?
- Given a vector field in two dimensions, how can we test whether or not it is conservative?

20.1 The story so far in 2D.

The following diagram summarises the relationships between conservative vector fields, path independent line integrals and closed line integrals we have seen so far.

$\boldsymbol{F} = \nabla f$.

$$
\begin{tikzcd}
 & \boldsymbol{F} \text{ conservative} \arrow[dl, shift left, "\text{* Fundamental thm for line integrals.}"] \\
\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r} \text{ path independent} \arrow[ur, shift left] \arrow[dr, shift left] & \\
 & \displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0, \ \underline{\underline{\forall \text{ closed}}}\ C \arrow[ul, shift left]
\end{tikzcd}
$$

20.2 Clairaut's theorem and consequences (Stewart p921)

Suppose a function of two variables f is defined on a disc D that contains the point (a, b). If the functions $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are both continuous on D, then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

(proof p A46)

Say we have a conservative vector field $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j}$. This means that there exists an $f(x, y)$ such that

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}.$$

An immediate consequence of Clairaut's theorem is that

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial F_2}{\partial x}.$$

In otherwords, we have the following:

$$\boxed{\text{If } \boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} \text{ is a conservative vector field, then } \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.}$$

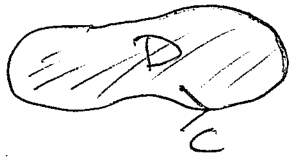
Let's add this to our diagram:

$$\begin{tikzcd}
& \boldsymbol{F} \text{ conservative} \arrow[dl, shift left] \arrow[dr] & \\
\displaystyle\int_C \boldsymbol{F} \cdot \boldsymbol{dr} \text{ path independent} \arrow[ur, shift left] \arrow[dr, shift left] & & \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x} \\
& \displaystyle\oint_C \boldsymbol{F} \cdot \boldsymbol{dr} = 0, \ \forall \text{ closed } C \arrow[ul, shift left] &
\end{tikzcd}$$

If we can reverse the new arrow, then we would have the criterion that we need! That is, the condition

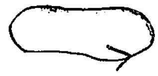
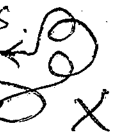
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

would be a test for a conservative vector field. To do this, we require one more piece of the puzzle. That is Green's theorem.



doesn't intersect itself anywhere between endpoints.

20.3 Green's theorem

 simple  X.

Let D be a region in the x-y plane bounded by a piecewise-smooth, simple closed curve C, which is traversed with D always on the left. Let $F_1(x, y)$, $F_2(x, y)$, $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ be continuous in D. Then "positively oriented"

$$\boxed{\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\, dx\, dy = \oint_C (F_1 dx + F_2 dy).}$$

proof p1092 Stewart D is type I or II.

This theorem relates a double integral to a line integral over a closed curve. For example, we can use Green's theorem to evaluate complicated line integrals by treating them as double integrals, or vice versa.

Regarding our discussion on conservative vector fields, we have the following *corollary* to Green's theorem:

$$\text{If } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \text{ then } \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0.$$

Note that $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j}$.

If we add this to our diagram, we can now link any four statements via the arrows. In otherwords all four statements are equivalent.

$$\begin{tikzcd}
& \boxed{\boldsymbol{F} \text{ conservative}} \arrow[dl, shift left] \arrow[dr, "\text{*Clairaut's theorem}"] & \\
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} \text{ path independent} \arrow[ur, shift left] \arrow[dr, shift left] & & \boxed{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}} \arrow[dl, "\text{*Green's theorem}"] \\
& \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0, \boxed{\forall \text{ closed } C} \arrow[ul, shift left] &
\end{tikzcd}$$

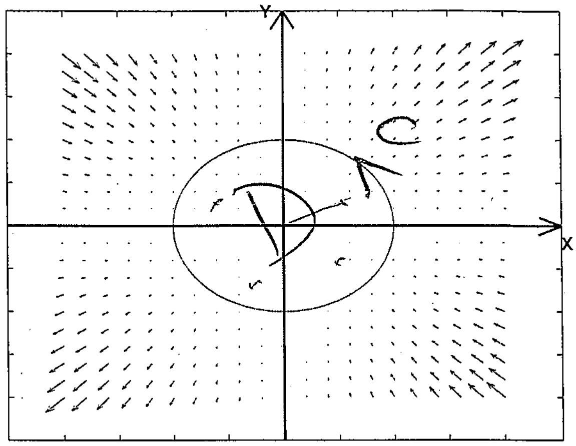
In particular, we now have a test to determine whether or not a given two dimensional vector field is conservative:

$$\text{The vector field } \boldsymbol{F} \text{ is conservative if and only if } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

$A \Rightarrow B$, "not B" $\Rightarrow$ "not A"

contrapositive

20.3.1 Find the work done by the force $F = x^2yi + xy^2j$ anticlockwise around the circle with centre at the origin and radius a.



Note conditions of Green's theorem are met.

$$\text{work} = \oint_C \underline{F}.d\underline{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

polar coords: $x = r\cos\theta$, $y = r\sin\theta$.

$$D = \{(r,\theta) \mid 0 \le r \le a, \ 0 \le \theta \le 2\pi\}.$$

$$\frac{\partial F_2}{\partial x} = y^2, \quad \frac{\partial F_1}{\partial y} = x^2$$

$$y^2 - x^2 = r^2(\sin^2\theta - \cos^2\theta)$$

$$= -r^2\cos 2\theta \quad \text{(trig. ident.)}$$

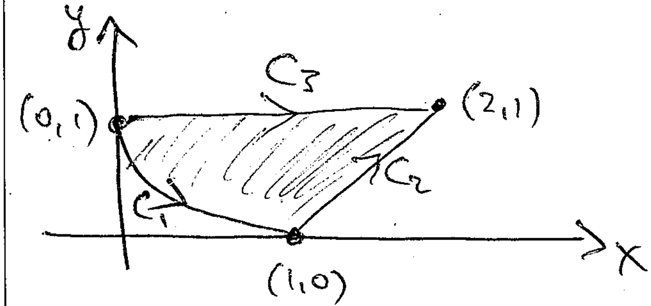
$$\Rightarrow \oint_C \underline{F}.d\underline{r} = -\int_0^a \int_0^{2\pi} r^2\cos 2\theta \, . \, r \, d\theta \, dr$$

$$= 0$$

Note, however, $\underline{F}$ is NOT conservative!

$\underline{F} = 2xy\,\underline{i} + (x^2+3y^2)\,\underline{j}$

20.3.2 Evaluate the line integral $\int_C 2xy\ dx + (x^2+3y^2)\ dy$, where C is the path from $(0,1)$ to $(1,0)$ along $y=(x-1)^2$ and then from $(1,0)$ to $(2,1)$ along $y=x-1$.



$$\left.\begin{array}{l} F_1 = 2xy \Rightarrow \dfrac{\partial F_1}{\partial y} = 2x \\ F_2 = x^2+3y^2 \Rightarrow \dfrac{\partial F_2}{\partial x} = 2x \end{array}\right\} \Rightarrow \underline{F} \text{ is conservative.}$$

& note that $F_1, F_2, \frac{\partial F_1}{\partial y}, \frac{\partial F_2}{\partial x}$ are cts everywhere.

$\Rightarrow \int_C \underline{F}\cdot d\underline{r}$ are path independent.

$\Rightarrow$ can choose any path from $(0,1)$ to $(2,1)$.

C_3: $\underline{r}(t) = t\,\underline{i} + 1\,\underline{j} \quad 0 \le t \le 2.$

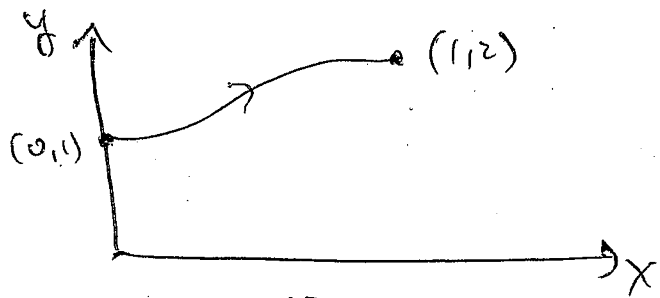
$$\Rightarrow \int_{C_3} \underline{F}\cdot d\underline{r} = \int_0^2 \left(\underline{F}\cdot\frac{d\underline{r}}{dt}\right) dt.$$

$\left(\text{Note } \frac{d\underline{r}}{dt} = \underline{i}\right)$

$$= \int_0^2 2t\ dt = 4.$$

$$\underline{F} = (3+2xy)\underline{i} + (x^2-3y^2)\underline{j}$$

20.3.3 **Evaluate** $\displaystyle\int_C (3+2xy)dx + (x^2-3y^2)dy$ **where** C **is the curve parametrised by** $\boldsymbol{r}(t) = (1-\cos(\pi t))\boldsymbol{i} + (1+\sin^3(\pi t))\boldsymbol{j}$ **for** $0 \le t \le 1/2$.



$$F_1 = 3 + 2xy \Rightarrow \frac{\partial F_1}{\partial y} = 2x$$

$$F_2 = x^2 - 3y^2 \Rightarrow \frac{\partial F_2}{\partial x} = 2x$$

$\Rightarrow \underline{F}$ is conservative.

Same $\underline{F}$ as p114.

we know $\underline{F} = \underline{\nabla} f$

$f(x,y) = 3x + x^2y - y^3 + c$ (from p114)

Fund. thm. (p120)

$$\Rightarrow \int_C \underline{F}\cdot d\underline{r} = f(1,2) - f(0,1)$$

$$= 3 + 2 - 8 + c - (-1 + c)$$

$$= -2.$$