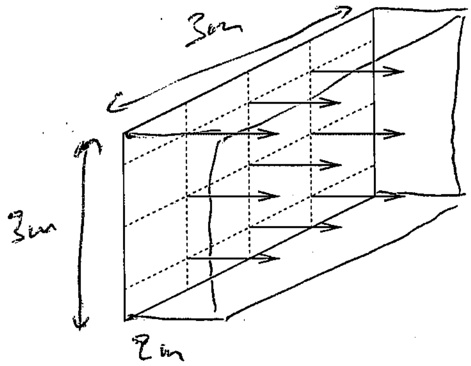
21 Flux of a vector field

By the end of this section, you should be able to answer the following questions:

- What is the flux of a constant vector field across a flat surface in 3D?
- What is the flux of a vector field across a plane curve in 2D?

In this section we introduce the concept of *flux*: In three dimensions, the flux of a vector field across a given surface is defined to be the "flow rate" of the vector field through the surface.

Since many vector fields involve no motion (eg. electric fields, magnetic fields), this definition can be very difficult to comprehend at first. A nice context for working with flux in order to understand its definition is by considering the velocity vector of a fluid (so now we do have motion). In three dimensions, the flux of a fluid across a surface is given in units of volume per unit time. In other words, the flux tells us how much of the fluid (volume) passes through a given surface in one second.

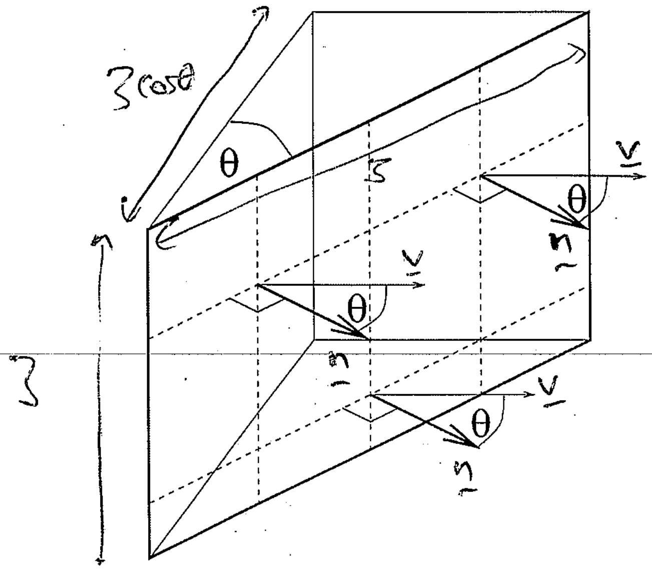


Consider a river flowing at a constant velocity of $2m/s$ in only one direction. Now imagine placing a $3m$ square fishing net into the river so that it somehow stays perpendicular to the flow of the river. What is the flux of the water through the net?

Volume through net after 1 sec

$= 3 \times 3 \times 2 = 18\,m^3$

$\Rightarrow$ flux $= 18\,m^3/sec.$



Now if we rotate the net through an angle θ, what is the flux through the net?

In 1 sec, same vol. passes through this net as net perpendicular to motion with dimension $3 \times 3\cos\theta$.
$\Rightarrow$ vol. after 1 sec $= 3 \times 3\cos\theta \times 2 = 18\cos\theta\, m^3$
$\Rightarrow$ flux $= 18\cos\theta\ m^3/sec$.

OR. Component of flow perp. to net is $\underset{\sim}{v} \cdot \underset{\sim}{n}$ ($\underset{\sim}{n}$ unit vector perp. to net = "unit normal")

$\underset{\sim}{v} \cdot \underset{\sim}{n} = |\underset{\sim}{v}||\underset{\sim}{n}|\cos\theta = 2 \times 1 \times \cos\theta$.

$\Rightarrow$ flux $= \underset{\sim}{v} \cdot \underset{\sim}{n} \times$(area of net)

$= 2\cos\theta \times 3 \times 3 = 18\cos\theta$.

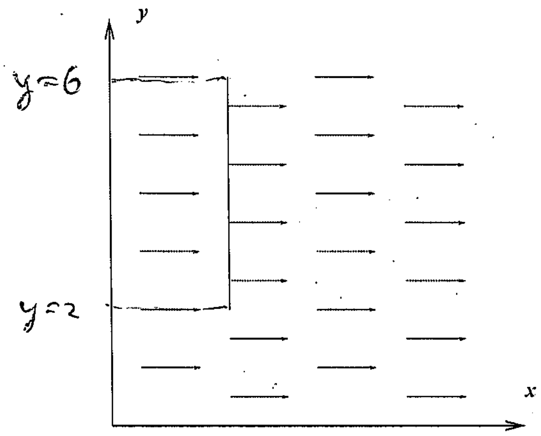
21.1 Flux in 2D

Before we look at the flux of a vector field through more general surfaces, let's look at flux in two dimensions, by considering the flow of a two dimensional fluid through a curve in the x-y plane. Note that in this context of a fluid in 2D, flux has dimensions *area* per unit time.

velocity $\underline{v}$

To start, consider the problem of calculating the flux of a fluid with constant velocity $v = 2i$ through a line segment C perpendicular to the flow, where C is given by

$$C = \{(x, y) \mid x = 2,\ 2 \le y \le 6\}.$$

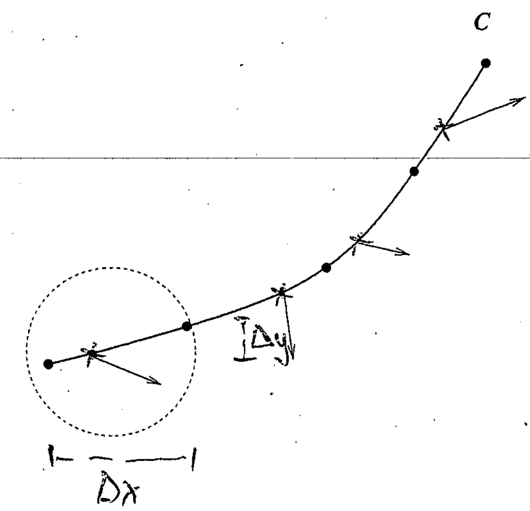


$$\text{flux} = (\underline{v} \cdot \underline{n}) \times (\text{length of } C)$$
$$= 2 \times 4 = 8\ m^2/\text{sec}.$$

cts vector function.

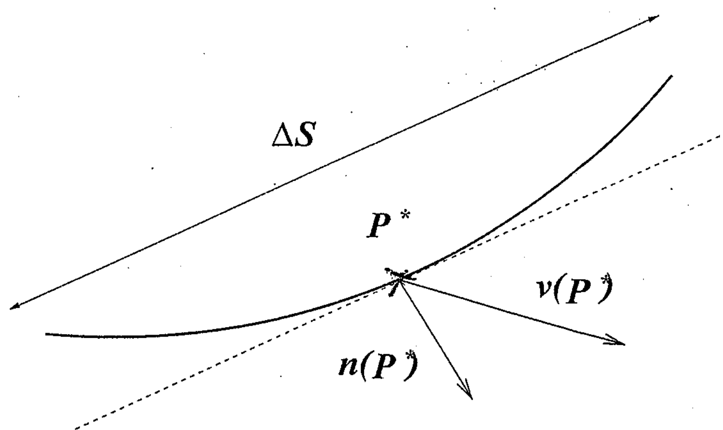
Now consider calculating the flux of the velocity vector $\boldsymbol{v}(x, y)$ in the x-y plane through a curve C. (parametric curve)

We first divide C up into arcs of length ΔS, and *approximate* $\boldsymbol{v}$ as constant over each arc.



This constant vector over each arc shall be evaluated at a representative point in each arc, say $P^* = (x^*, y^*)$. We also approximate the arc as a straight line, so that

$$\Delta S \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \approx |\boldsymbol{r}'(t)|\ \Delta t.$$



The component of $\boldsymbol{v}$ which is perpendicular to C (over ΔS) is $\approx \boldsymbol{v}(P^*) \cdot \boldsymbol{n}(P^*)$. We then have

$$\text{flux through one arc } \approx \left(\boldsymbol{v}(P^*) \cdot \boldsymbol{n}(P^*)\right)\Delta S.$$

$$\Rightarrow \text{ total flux through } C \approx \sum_i \boldsymbol{v}(P_i^*) \cdot \boldsymbol{n}(P_i^*)\Delta S_i.$$

If we take the limit as $\Delta S \to 0$, we obtain an exact expression for the flux over the entire curve C as a line integral:

$$\text{Flux} = \int_C (\boldsymbol{v} \cdot \boldsymbol{n})\, dS,$$

(line integral in 2D $\int_C f(x,y)\,dS$)

where $\boldsymbol{n}$ is a unit vector normal to C.

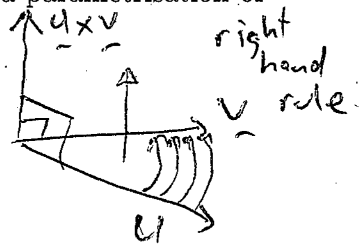
We use this expression as a *definition* of flux of any two dimensional vector field $\boldsymbol{v}$ across a plane curve C. Note then that

$$\text{dimensions of flux (in 2D)} = (\text{dimensions of } \boldsymbol{v}) \times (\text{distance}).$$

21.1.1 Evaluating flux in 2D

To evaluate the line integral in the definition of flux, we need a parametrisation of C, say $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$ for $a \le t \le b$ (say). We define

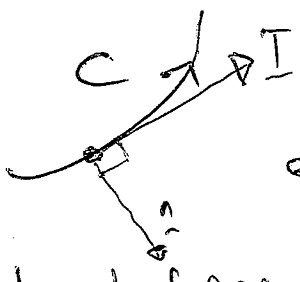
$$\boldsymbol{r}'(t) = \dot{x}\ \boldsymbol{i} + \dot{y}\ \boldsymbol{j}.$$



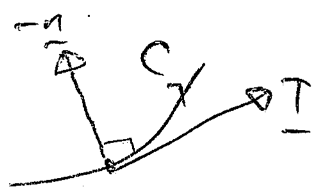
A unit tangent vector to C is then given by

$$\boldsymbol{T} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}.$$

By the definition of vector cross product, and since $\boldsymbol{k}$ is a unit vector normal to the x-y plane, being careful of the direction of $\boldsymbol{n}$, we can take



$$\boldsymbol{n} = \boldsymbol{T} \times \boldsymbol{k} = \frac{1}{|\boldsymbol{r}'(t)|}\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dot{x} & \dot{y} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$



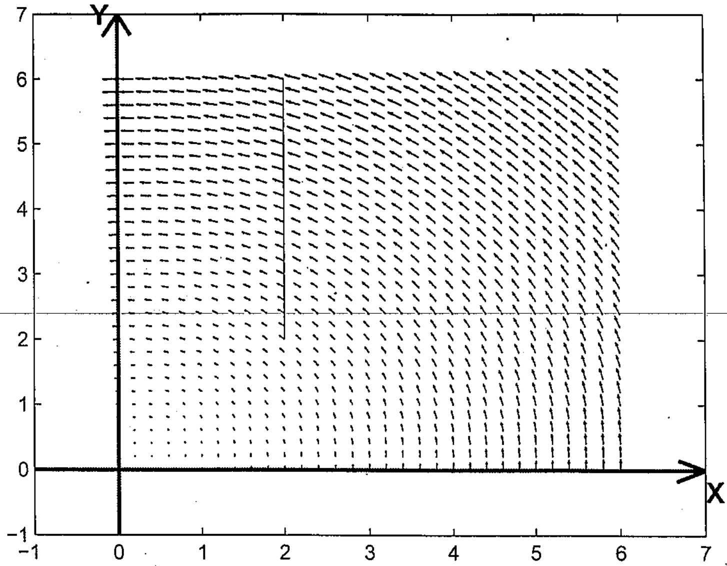
$-\underline{n} = \underline{k} \times \underline{T}$

$$\begin{aligned} &= \frac{1}{|\boldsymbol{r}'(t)|}(\dot{y}\ \boldsymbol{i} - \dot{x}\ \boldsymbol{j}) \\ \Rightarrow \boldsymbol{v} \cdot \boldsymbol{n} &= \frac{\boldsymbol{v} \cdot (\dot{y}\ \boldsymbol{i} - \dot{x}\ \boldsymbol{j})}{|\boldsymbol{r}'(t)|} \\ &= \frac{v_1 \dot{y} - v_2 \dot{x}}{|\boldsymbol{r}'(t)|}, \end{aligned}$$

where $\boldsymbol{v}(x,y) = v_1(x,y)\boldsymbol{i} + v_2(x,y)\boldsymbol{j}$. Noting also that in the integral we have $\boxed{dS = |\boldsymbol{r}'(t)|\ dt,}$ we then have a means of evaluating the line integral (2D flux integral) as

$$\int_C \boldsymbol{v} \cdot \boldsymbol{n}\ dS = \int_{t=a}^{t=b} (v_1(t)\dot{y} - v_2(t)\dot{x})\ dt.$$

important: $\underline{n}$ determines the direction of positive flux.



C is parallel to y-axis

$\Longrightarrow \underline{n} = \underline{i}$

21.1.2 Calculate the flux of $v = -yi + xj$ (in the positive x direction) across the line $x = 2$ (for $2 \le y \le 6$). C

flux $= \int_C \underline{v} \cdot \underline{n} \, ds$

$C: \underline{r}(t) = 2\underline{i} + t\underline{j}$, $\quad 2 \le t \le 6 \Rightarrow \underline{r}'(t) = \underline{j}$

$\underline{v} \cdot \underline{n} = (-y\underline{i} + x\underline{j}) \cdot \underline{i} = -y = -t$.

in terms of t.

Also, in $\int$, $ds = |\underline{r}'(t)| dt = dt$.

$\Rightarrow$ flux $= -\int_2^6 t \, dt = -\frac{1}{2} t^2 \Big|_2^6 = -16$.

(if $\underline{v}$ is velocity, in SI units, flux is $\frac{m}{s} \cdot m = \frac{m^2}{s}$)

Answer is negative! See from graph, $\underline{v}$ is "flowing" right to left across C, but direction of positive flux was specified as left to right.

21.2 Outward flux across a closed curve in the plane

Let C be a piecewise-smooth, simple closed curve. Let $v_1(x,y)$, $v_2(x,y)$ be continuous in the region bounded by C. (Note that these are some of the conditions of Green's theorem!)

The net outward flux of $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j}$ across C is given by

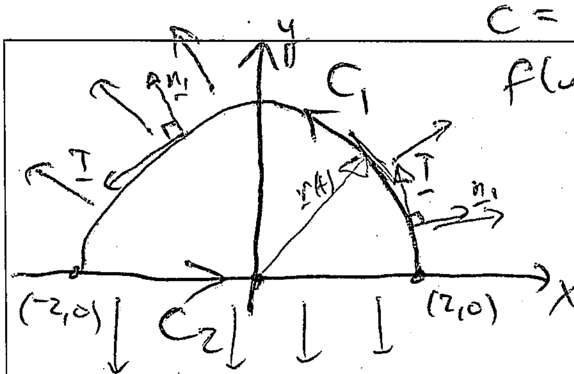
$$\text{Net outward flux } = \oint_C \boldsymbol{v}\cdot\boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is a unit vector normal to C, directed outward from the region bounded by C.

positive direction.

21.2.1 Calculate the outward flux of $v = xy\boldsymbol{i} + xy\boldsymbol{j}$ across the curve from (2,0) to (-2,0) via the semicircle of radius 2 centred at the origin (for $y \geq 0$) followed by the straight line from (-2,0) to (2,0).

$C = C_1 \cup C_2$



$$\text{flux} = \oint_C \underset{\sim}{v} \cdot \underset{\sim}{n}\, ds.$$
$$= \int_{C_1} (\underset{\sim}{v}\cdot \underset{\sim}{n_1})\, ds + \int_{C_2} \underset{\sim}{v}\cdot \underset{\sim}{n_2}\, dS$$

C_1: $\underset{\sim}{r}(t) = 2\cos t\, \underset{\sim}{i} + 2\sin t\, \underset{\sim}{j}$, $0 \leq t \leq \pi$.

$\underset{\sim}{r}'(t) = -2\sin t\, \underset{\sim}{i} + 2\cos t\, \underset{\sim}{j}$, $|\underset{\sim}{r}'(t)| = 2$

outward unit normal vector n_1

$$\underset{\sim}{n_1} = \frac{\underset{\sim}{r}'(t)}{|\underset{\sim}{r}'(t)|} \times \underset{\sim}{k} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j}$$

$$\underset{\sim}{v}\cdot\underset{\sim}{n_1} = (4\cos t \sin t\, \underset{\sim}{i} + 4\cos t\sin t\, \underset{\sim}{j})\cdot(\cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j})$$
$$= 4\cos^2 t\sin t + 4\cos t\sin^2 t$$

outward flux across C_1

$$= \int_{C_1} \underline{v} \cdot \underline{n}_1 \, ds \qquad \left(ds = |\underline{r}'(t)| dt = 2\, dt \right)$$

$$= \int_0^{\pi} 2 \left(4 \cos^2 t \sin t + 4 \cos t \sin^2 t \right) dt$$

$$= \cdots = \frac{16}{3}.$$

For C_2, note that $\underline{v} = \underline{0}$ $(y=0)$

$\Rightarrow$ flux across C_2 $= 0$

net outward flux across C

$$= \oint_C \underline{v} \cdot \underline{n} \, ds = \int_{C_1} \underline{v} \cdot \underline{n}_1 \, ds + \int_{C_2} \underline{v} \cdot \underline{n}_2 \, ds$$

$$= \frac{16}{3} + 0$$