22 Divergence of a vector field (div)

By the end of this section, you should be able to answer the following questions:

- How do you calculate the divergence of a given vector field?
- What is the significance of divergence?
- How does it relate to flux?

In this section we introduce the concept of *divergence* of a vector field.

22.1 Calculating divergence

$\underline{\nabla} = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$

Let

$$\boldsymbol{v}(x, y, z) = v_1(x, y, z)\boldsymbol{i} + v_2(x, y, z)\boldsymbol{j} + v_3(x, y, z)\boldsymbol{k}$$

be a differentiable vector function. Then the function

$$\text{div } \boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \nabla \cdot \boldsymbol{v}$$

is called the divergence of $\boldsymbol{v}$. Note div $\boldsymbol{v}$ is a scalar quantity.

Divergence has an analogous definition in two dimensions. For

$$\boldsymbol{F}(x, y) = F_1(x, y)\boldsymbol{i} + F_2(x, y)\boldsymbol{j} \quad \Rightarrow \quad \text{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

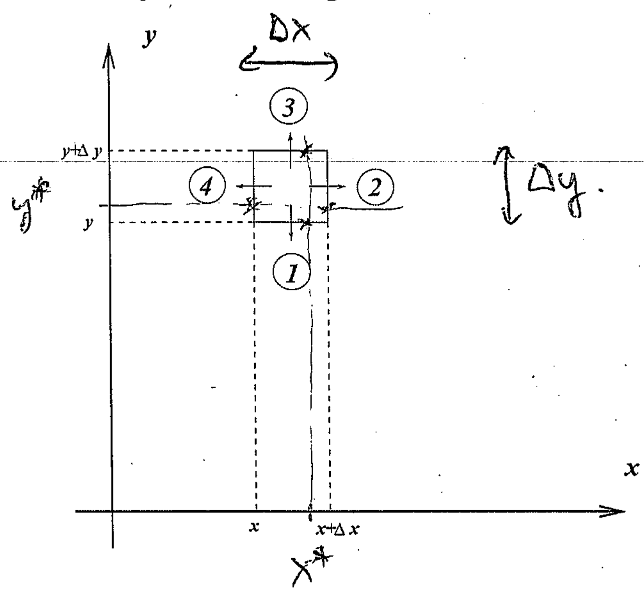
22.1.1 Example: $v = xy^2 i + xyzj + yz^2 k$. Find div v

$$\underline{\nabla} \cdot \underline{v} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(yz^2)$$

$$= y^2 + xz + 2yz.$$

22.2 Understanding div in two dimensions.

Consider the flow of a two dimensional fluid with continuous velocity field $\boldsymbol{v}(x, y) = v_1(x, y)\boldsymbol{i} + v_2(x, y)\boldsymbol{j}$. Our aim is to calculate the outward flux from a small rectangle in the plane of area $\Delta x \Delta y$ as in the diagram below.



We first approximate the flux across each of the four sides of the rectangle. In each case the approximation will be $\boldsymbol{v} \cdot \boldsymbol{n}\Delta S$, where we assume $\boldsymbol{v}$ is constant over each edge. Also let $x^* \in [x, x + \Delta x]$ and $y^* \in [y, y + \Delta y]$ represent chosen points in each interval.

Edge 1: we evaluate $\boldsymbol{v}$ at (x^*, y) and assume it is constant across the entire edge. An outwardly pointing unit normal vector is $-\boldsymbol{j}$.

$$\text{flux } \approx \boldsymbol{v}(x^*, y) \cdot (-\boldsymbol{j})\ \Delta x.$$

Edge 2: we evaluate $\boldsymbol{v}$ at $(x + \Delta x, y^*)$ and assume it is constant across the entire edge. An outwardly pointing unit normal vector is $\boldsymbol{i}$.

$$\text{flux } \approx \boldsymbol{v}(x + \Delta x, y^*) \cdot (\boldsymbol{i})\ \Delta y.$$

Edge 3: we evaluate $\boldsymbol{v}$ at $(x^*, y + \Delta y)$ and assume it is constant across the entire edge. An outwardly pointing unit normal vector is $\boldsymbol{j}$.

$$\text{flux } \approx \boldsymbol{v}(x^*, y + \Delta y) \cdot (\boldsymbol{j})\ \Delta x.$$

Edge 4: we evaluate $\boldsymbol{v}$ at (x, y^*) and assume it is constant across the entire edge. An outwardly pointing unit normal vector is $-\boldsymbol{i}$.

$$\text{flux } \approx \boldsymbol{v}(x, y^*) \cdot (-\boldsymbol{i})\ \Delta y.$$

add all 4 terms . . .

Combining all four terms gives an approximation to the net outward flux:

net outward flux

$$
\begin{aligned}
&\approx (\boldsymbol{v}(x+\Delta x, y^*) - \boldsymbol{v}(x, y^*)) \cdot \boldsymbol{i}\ \Delta y + (\boldsymbol{v}(x^*, y+\Delta y) - \boldsymbol{v}(x^*, y)) \cdot \boldsymbol{j}\ \Delta x \\
&= \left(\frac{\boldsymbol{v}(x+\Delta x, y^*) - \boldsymbol{v}(x, y^*)}{\Delta x} \right) \cdot \boldsymbol{i} \Delta x \Delta y + \left(\frac{\boldsymbol{v}(x^*, y+\Delta y) - \boldsymbol{v}(x^*, y)}{\Delta y} \right) \cdot \boldsymbol{j} \Delta x \Delta y \\
&= \left(\frac{v_1(x+\Delta x, y^*) - v_1(x, y^*)}{\Delta x} + \frac{v_2(x^*, y+\Delta y) - v_2(x^*, y)}{\Delta y} \right) \Delta x \Delta y \\
&\approx \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) \Delta x \Delta y. \quad \Bigg\} \simeq \text{ at } (x^*, y^*) \\
&= \operatorname{div}(\boldsymbol{v}) \Delta x \Delta y.
\end{aligned}
$$

Hence, we have

$$\frac{\text{flux out of a rectangle}}{\text{area of rectangle}} \approx \operatorname{div}(\boldsymbol{v}).$$

If we take the limit as the dimensions of the rectangle appproach 0, we have

$$\boxed{\operatorname{div}(\boldsymbol{v}) = \lim_{\Delta A \to 0} \frac{\text{flux out of } \Delta A}{\Delta A}}$$

In other words, $\operatorname{div}(\boldsymbol{v})$ is the "outward flux density" of $\boldsymbol{v}$ at a given point.

This concept generalises quite naturally to three dimensions:

$$\operatorname{div}(\boldsymbol{v}(x, y, z)) = \lim_{\Delta V \to 0} \frac{\text{flux out of } \Delta V}{\Delta V}.$$

In the context of fluids (our main focus so far) we can say $\operatorname{div}(\boldsymbol{v}(x, y, z))$ measures the tendency of the fluid to "diverge" from the point (x, y, z).

Compare with 2D mass density

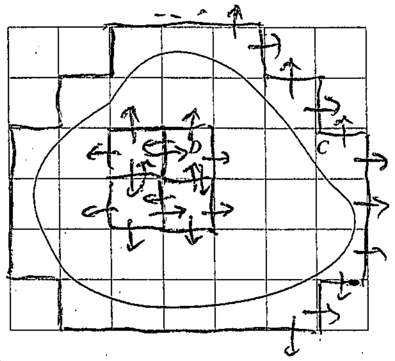
$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\text{mass of } \Delta A}{\Delta A} \qquad (p88)$$

22.3 Outward flux across a closed curve in the plane (revisited)

One final calculation uses the divergence to calculate the net outward flux of $\boldsymbol{v}$ across a closed curve. We have already seen that we can evaluate this quantity by calculating $\oint_C \boldsymbol{v} \cdot \boldsymbol{n} \, dS$.

Now let D be a region in the x-y plane bounded by a piecewise-smooth, simple closed curve C, which is traversed with D always on the left. Let $v_1(x, y)$, $v_2(x, y)$ have continuous derivatives in D (again the conditions of Green's theorem!).

$\uparrow\downarrow \approx 0$



only surviving terms are on the boundary.

By the previous calculation involving divergence, we can also approximate the outward flux from the region by dividing D up into small rectangles and approximating the net outward flux across each rectangle. We know that for one rectangle,

$$\text{outward flux of one rectangle} \approx \operatorname{div}(\boldsymbol{v}(x^*, y^*))\Delta x \Delta y,$$

where (x^*, y^*) is some point inside the rectangle. We repeat this for each rectangle containing part of the region D, so that

$$\text{net outward flux across } C \approx \sum \operatorname{div}(\boldsymbol{v}(x^*, y^*))\Delta x \Delta y.$$

Taking the limit as $\Delta x, \Delta y \to 0$, we have

$$\text{net outward flux across } C = \iint_D \operatorname{div}(\boldsymbol{v}(x, y)) \, dA,$$

the double integral of the region D.

To obtain the flux, we integrate the flux density over the region. Compare this with the context of mass density: to obtain the mass, we integrate the mass density over the region.

Finally, the two ways of calculating the same quantity must obviously be equal:

$$\boxed{\oint_C \boldsymbol{v}(x, y) \cdot \boldsymbol{n} \, dS = \iint_D \operatorname{div}(\boldsymbol{v}(x, y)) \, dA.}$$

"Flux form of Green's theorem".

22.4 Relationship to Green's theorem

We have seen how to evaluate the 2D flux integral:

$$\oint_C \boldsymbol{v} \cdot \boldsymbol{n} \; dS = \int_{t=a}^{t=b} (v_1(t)\dot{y} - v_2(t)\dot{x}) \; dt.$$

This can be rewritten as

$\underline{v} = v_1 \underline{i} + v_2 \underline{j}$

$$\oint_C \boldsymbol{v} \cdot \boldsymbol{n} \; dS = \oint_C \underbrace{v_1}_{F_2} \, dy \overbrace{- v_2}^{F_1} \, dx.$$

$\underline{F} = -v_2 \underline{i} + v_1 \underline{j}$

If we define $F_1(x, y) = -v_2(x, y)$ and $F_2(x, y) = v_1(x, y)$, we then have

$$\oint_C \boldsymbol{v} \cdot \boldsymbol{n} \; dS = \oint_C F_1 \; dx + F_2 \; dy.$$

We also have

$$\operatorname{div}(\boldsymbol{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y},$$

so that

$$\iint\limits_D \operatorname{div}(\boldsymbol{v}) \; dA = \iint\limits_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \; dA.$$

This tells us that in terms of the new vector field

$$\boldsymbol{F} = -v_2 \boldsymbol{i} + v_1 \boldsymbol{j} = F_1 \boldsymbol{i} + F_2 \boldsymbol{j},$$

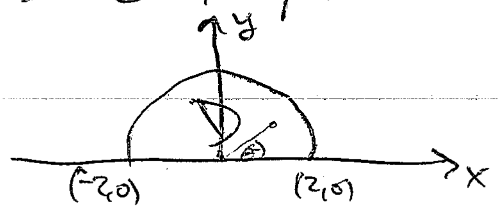
the two ways of calculating flux are given by → flux of $\underline{v}$, not $\underline{F}$.

$$\iint\limits_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \; dx \; dy = \oint_C (F_1 dx + F_2 dy).$$

This is none other than Green's theorem. So the flux identity we obtained at the bottom of the previous page is just Green's theorem in disguise. We shall call this the *flux form* of Green's theorem.

22.4.1 Use the flux form of Green's theorem to calculate the outward flux of $v = xyi + xyj$ across the curve from (2,0) to (-2,0) via the semicircle of radius 2 centred at the origin (for $y \geq 0$) followed by the straight line from (-2,0) to (2,0).

Same example as p137.



Use polar coords

$x = r\cos\theta$

$y = r\sin\theta$.

$$D = \{(r,\theta) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}.$$

$$\underline{\nabla} \cdot \underline{v} = \underbrace{\frac{\partial}{\partial x}(xy)}_{\underline{i}\text{ comp.}} + \underbrace{\frac{\partial}{\partial y}(xy)}_{\underline{j}\text{ comp.}}$$

$$= y + x.$$

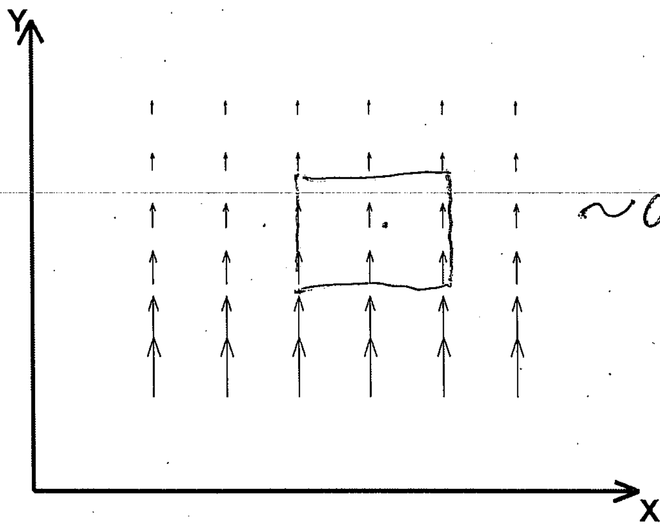
$\Rightarrow$ net outward flux $= \iint_D \underline{\nabla} \cdot \underline{v}\, dA$.

$$= \int_0^2 \int_0^\pi (r\cos\theta + r\sin\theta)\, r\, d\theta\, dr$$

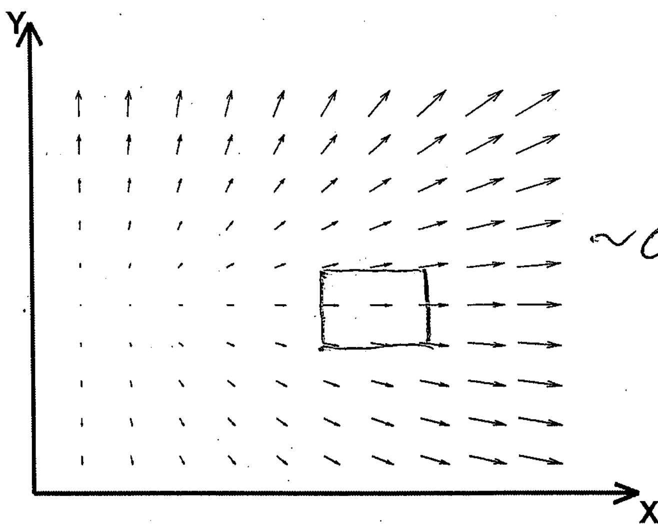
$$= \left(\int_0^2 r^2\, dr\right)\left(\int_0^\pi (\cos\theta + \sin\theta)\, d\theta\right)$$

$$= \frac{8}{3} \times 2 = 16/3.$$

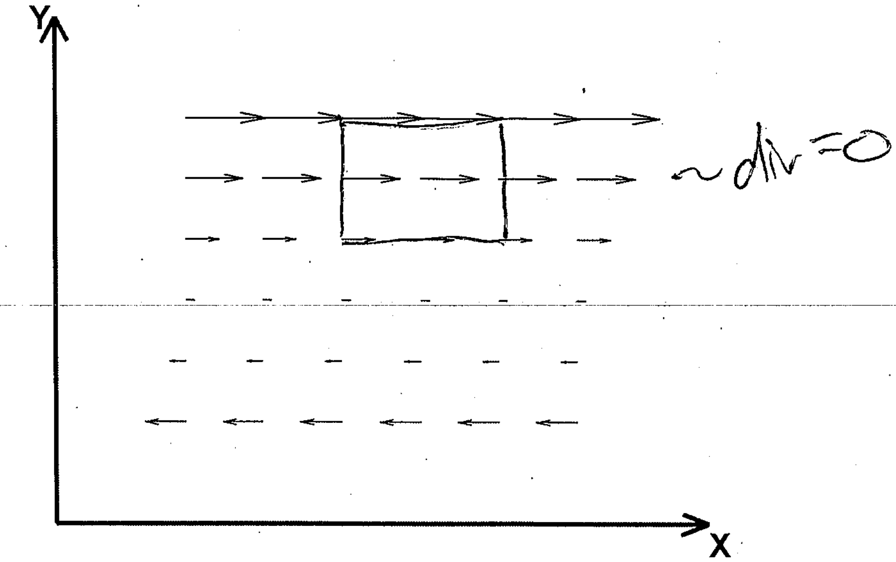
22.4.2 For the following graphs of vector fields, determine whether the divergence is positive, negative or zero.

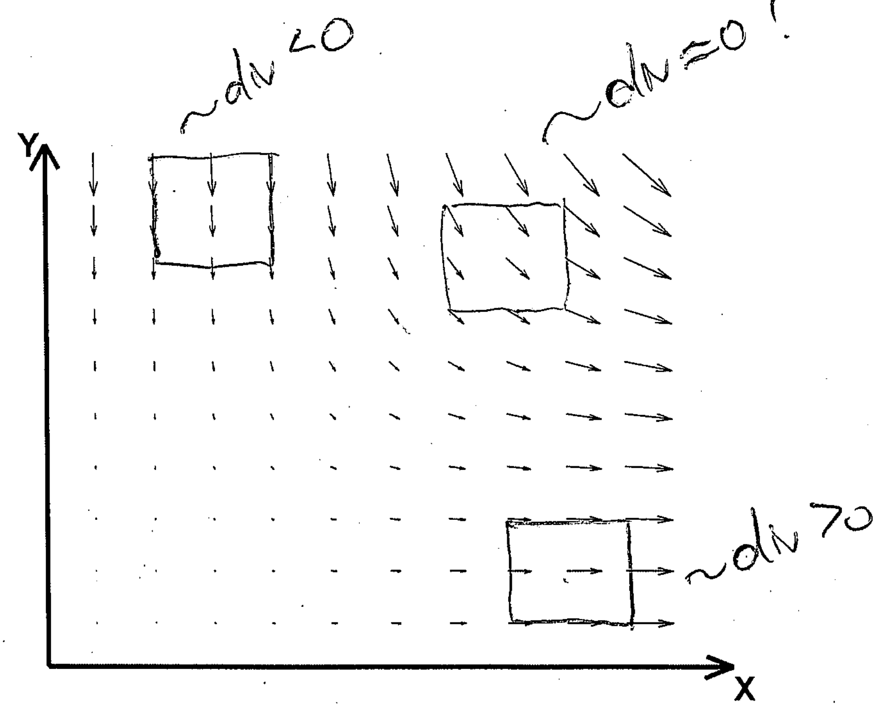


~div < 0



~div > 0


Y
~div=0
X


~div<0
~div=0?
Y
~div>0
X