# 23 Parametrisation of surfaces in $\mathbb{R}^3$

By the end of this section, you should be able to answer the following questions:

- What does it mean to parametrise a surface in $\mathbb{R}^3$?
- How do you parametrise certain surfaces?

## 23.1 Parametric surfaces

We have already seen two ways of representing a surface in $\mathbb{R}^3$: explicitly as $z = f(x, y)$ or implicitly as $F(x, y, z) = 0$.

Another way of representing a surface $S$ in $\mathbb{R}^3$ is by a parametrisation. This is where the coordinate variables are functions of two parameters $u$ and $v$:

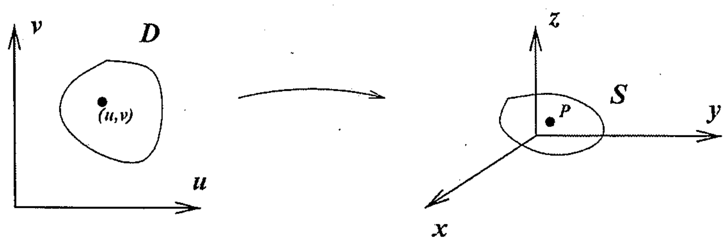
$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

and the vector

$$\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k}$$

traces out the surface as $u, v$ vary over some region $D$ in the "$u$-$v$ plane". So for every point $(u, v)$ in $D$, there corresponds a point on the surface $S$.

The following diagram shows the point $P$ on the surface $S$ which corresponds to the point $(u, v)$ in the region $D$ in the $u$-$v$ plane. As $(u, v)$ moves around all points in $D$, the point $P$ moves around in $S$, tracing out the entire surface.




Note that a surface defined explicitly by $z = f(x, y)$ is equivalent to a parametrisation

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + f(x, y)\boldsymbol{k},$$

where we treat the coordinate variables $x$ and $y$ as the parameters. Note that we have not specified any bounds on the variables. Often the challenge is to not only find suitable functions for a parametrisation, but for a finite surface to determine bounds on the parameters.

## 23.2 Parametrising surfaces using cylindrical and spherical coordinates

We can use our knowledge of cylindrical and spherical coordinates to parametrise certain surfaces with which these coordinates are naturally associated.

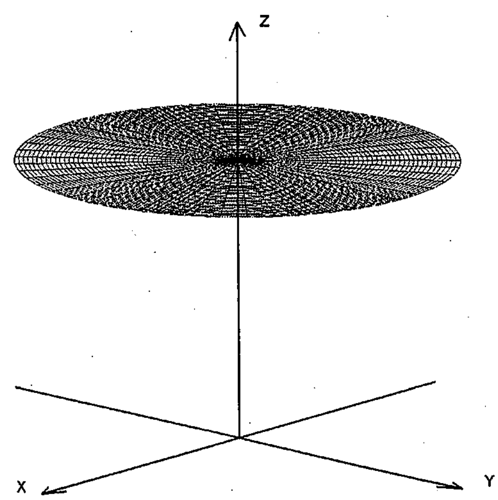
Recall cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Setting exactly one of the cylindrical coordinates to a constant value necessarily gives a parametric surface.

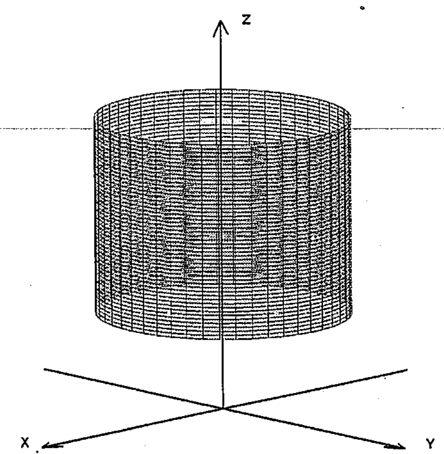
Setting $z = 2$ with $0 \le \theta \le 2\pi$, $0 \le r \le 3$ describes a disc of radius 3, centred at the $z$ axis lying in the plane $z = 2$:

$\underline{r}(r,\theta) = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + 2\,\underline{k}$.



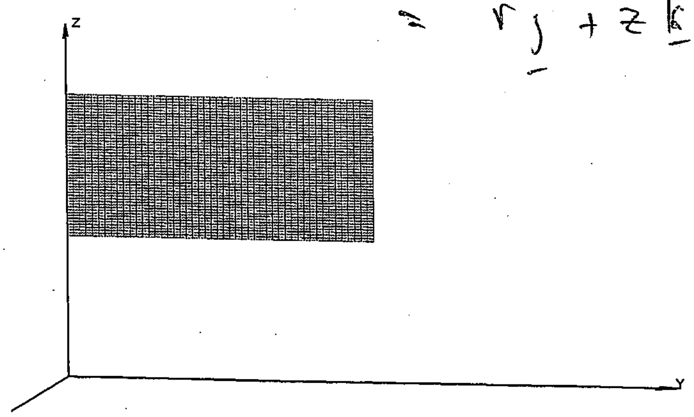
Setting $r = 5$ with $0 \leq \theta \leq 2\pi$, $1 \leq z \leq 3$ describes the surface of a cylinder of radius 5 and of height 2 between $z = 1$ and $z = 3$:

$$\underline{r}(\theta, z) = 5\cos\theta\, \underline{i} + 5\sin\theta\, \underline{j} + z\underline{k}.$$




Setting $\theta = \pi/2$ with $2 \leq z \leq 4$, $0 \leq r \leq 1$ describes a rectangle lying in the $y$-$z$ plane. Another description of the same surface would be $x = 0$, $\{(y, z) \mid 0 \leq y \leq 1,\ 2 \leq z \leq 4\}$:

$$\underline{r}(r, z) = r\cos\tfrac{\pi}{2}\, \underline{i} + r\sin\tfrac{\pi}{2}\, \underline{j} + z\underline{k}.$$
$$= r\underline{j} + z\underline{k}.$$



**23.2.1** Parametrise the paraboloid $z = 1 - x^2 - y^2$ for $z \geq 0$.

Based on cylindrical coords,

$x = r\cos\theta, \quad y = r\sin\theta,$

$z = 1 - (x^2 + y^2)$

$= 1 - r^2 \geq 0$

$\Longrightarrow \quad 0 \leq r \leq 1 \quad \& \quad 0 \leq \theta \leq \pi.$

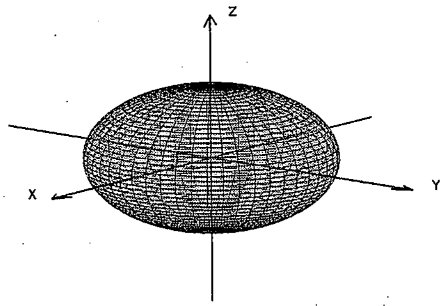
$\underline{r}(r, \theta) = r\cos\theta \, \underline{i} + r\sin\theta \, \underline{j} + (1 - r^2) \underline{k}$

Recall spherical coordinates: $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$.

Setting exactly one of the spherical coordinates to a constant value necessarily gives a parametric surface.

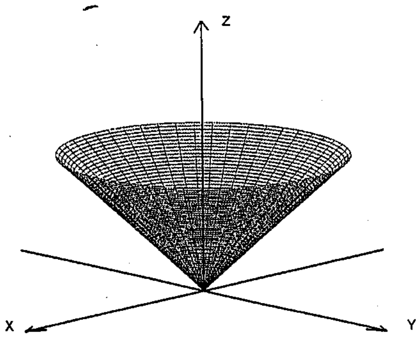
Setting $r = 2$ with $\underline{0 \le \theta \le 2\pi}$, $\underline{0 \le \phi \le \pi}$ describes the surface of a sphere of radius 2 centred at the origin:

$$\underline{r}(\theta, \phi) = 2\cos\theta\sin\phi\,\underline{i} + 2\sin\theta\sin\phi\,\underline{j} + 2\cos\phi\,\underline{k}.$$

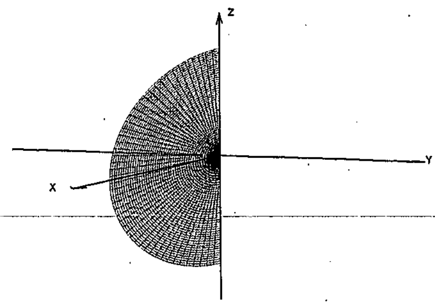



Setting $\phi = \pi/3$ with $0 \le r \le 2$, $0 \le \theta \le 2\pi$ describes the open cone with angle $\pi/3$ to the positive $z$-axis, the "mouth" of which lies on the sphere of radius 2 and with vertex located at the origin:

$$\begin{aligned}\underline{r}(r,\theta) &= r\cos\theta\sin\tfrac{\pi}{3}\,\underline{i} + r\sin\theta\sin\tfrac{\pi}{3}\,\underline{j} + r\cos\tfrac{\pi}{3}\,\underline{k}.\\ &= \frac{\sqrt{3}}{2}r\cos\theta\,\underline{i} + \frac{\sqrt{3}}{2}r\sin\theta\,\underline{j} + \frac{1}{2}r\,\underline{k}.\end{aligned}$$



Setting $\theta = 0$ with $0 \le r \le 3$, $0 \le \phi \le \pi$ describes the half disc of radius 3 lying in the $x$-$z$ plane: $\underline{r}(r, \phi) = \ldots$

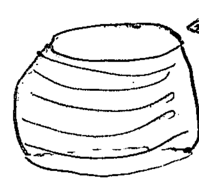



**23.2.2** **Parametrise the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = 2$ and $z = -2$.** (radius 4)

$x = 4\cos\theta \sin\phi$

$y = 4\sin\theta \sin\phi$

$z = 4\cos\phi$.




$0 \le \theta \le 2\pi$.

$z = 2 \Rightarrow 4\cos\phi = 2 \Rightarrow \cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$.

$z = -2 \Rightarrow 4\cos\phi = -2 \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$.

$\Rightarrow \frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$.

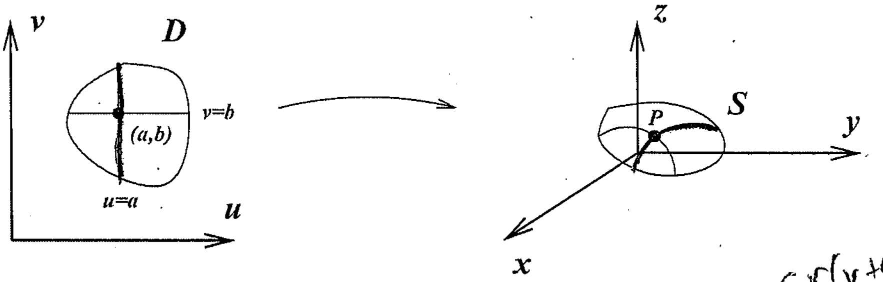
## 23.3 Tangent planes

Let $S$ be a surface parametrised by

$$\boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + z(u,v)\boldsymbol{k}.$$

Here we find the tangent plane to $S$ at a point $P$ specified by $\boldsymbol{r}(a,b)$.

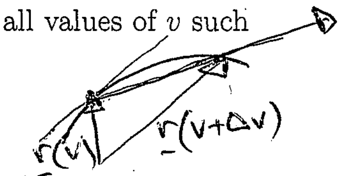
There are two important families of curves on $S$. One where $u$ is a constant, the other where $v$ is a constant. The diagram below shows the relationship between horizontal and vertical lines in $D$ (in the $u$-$v$ plane) and curves on $S$.




$\Delta v \ll 1$

$$\left(\frac{\underline{r}(v+\Delta v) - \underline{r}(v)}{\Delta v}\right)$$

Setting $u = a$ defines a curve on $S$ parametrised by $\boldsymbol{r}(a,v)$, for all values of $v$ such that $(a,v)$ lies in $D$. A tangent vector to this curve at $P$ is




tangent vector $$\underline{\boldsymbol{r}_v} = \frac{\partial x}{\partial v}(a,b)\boldsymbol{i} + \frac{\partial y}{\partial v}(a,b)\boldsymbol{j} + \frac{\partial z}{\partial v}(a,b)\boldsymbol{k}.$$

Similarly setting $v = b$ defines another curve on $S$ parametrised by $\boldsymbol{r}(u,b)$. A tangent vector to this curve at $P$ is

tangent vector $$\boldsymbol{r}_u = \frac{\partial x}{\partial u}(a,b)\boldsymbol{i} + \frac{\partial y}{\partial u}(a,b)\boldsymbol{j} + \frac{\partial z}{\partial u}(a,b)\boldsymbol{k}.$$

If $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are continuous and $\boldsymbol{r}_u \times \boldsymbol{r}_v$ is never $\boldsymbol{0}$ inside $D$ (we make an exception for points on the boundary of $D$), we call the surface *smooth* (it has no "kinks").

For a smooth surface, $\boldsymbol{r}_u \times \boldsymbol{r}_v$ is a normal vector at any point inside $D$. This vector evaluated at $(u,v) = (a,b)$ is also normal to the tangent plane at the point $P = (x(a,b), y(a,b), z(a,b))$.

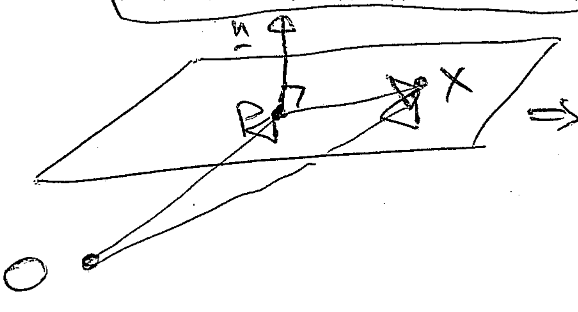
The equation of the tangent plane at $P$ is given by

$$\boxed{(\boldsymbol{r}_u(a,b) \times \boldsymbol{r}_v(a,b)) \cdot ((x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) - \boldsymbol{r}(a,b)) = 0.}$$

must have

$n \cdot \overline{PX} = 0$

$\Rightarrow \underline{n} \cdot (\overrightarrow{OX} - \overrightarrow{OP}) = 0$



**23.3.1** Find the tangent plane to the surface parametrised by $\boldsymbol{r}(u,v) = u^2\boldsymbol{i} + v^2\boldsymbol{j} + (u+2v)\boldsymbol{k}$ at the point $(1,1,3)$.

$\underline{r}_u = 2u\underline{i} + \underline{k}$

$\underline{r}_v = 2v\underline{j} + 2\underline{k}$.

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix}$$

$$= -2v\underline{i} - 4u\underline{j} + 4uv\underline{k}$$

point $(1,1,3)$ corresponds to $u=1, v=1$,

so $\underline{r}_u \times \underline{r}_v \Big|_{(1,1)} = -2\underline{i} - 4\underline{j} + 4\underline{k}$.

$\Rightarrow$ equ. of tangent plane

$(\underline{r}_u \times \underline{r}_v) \cdot (x\underline{i} + y\underline{j} + z\underline{k} - (\underline{i} + \underline{j} + 3\underline{k})) = 0$

$\Rightarrow -2(x-1) - 4(y-1) + 4(z-3) = 0$

$\Rightarrow \boxed{x + 2y - 2z + 3 = 0}$

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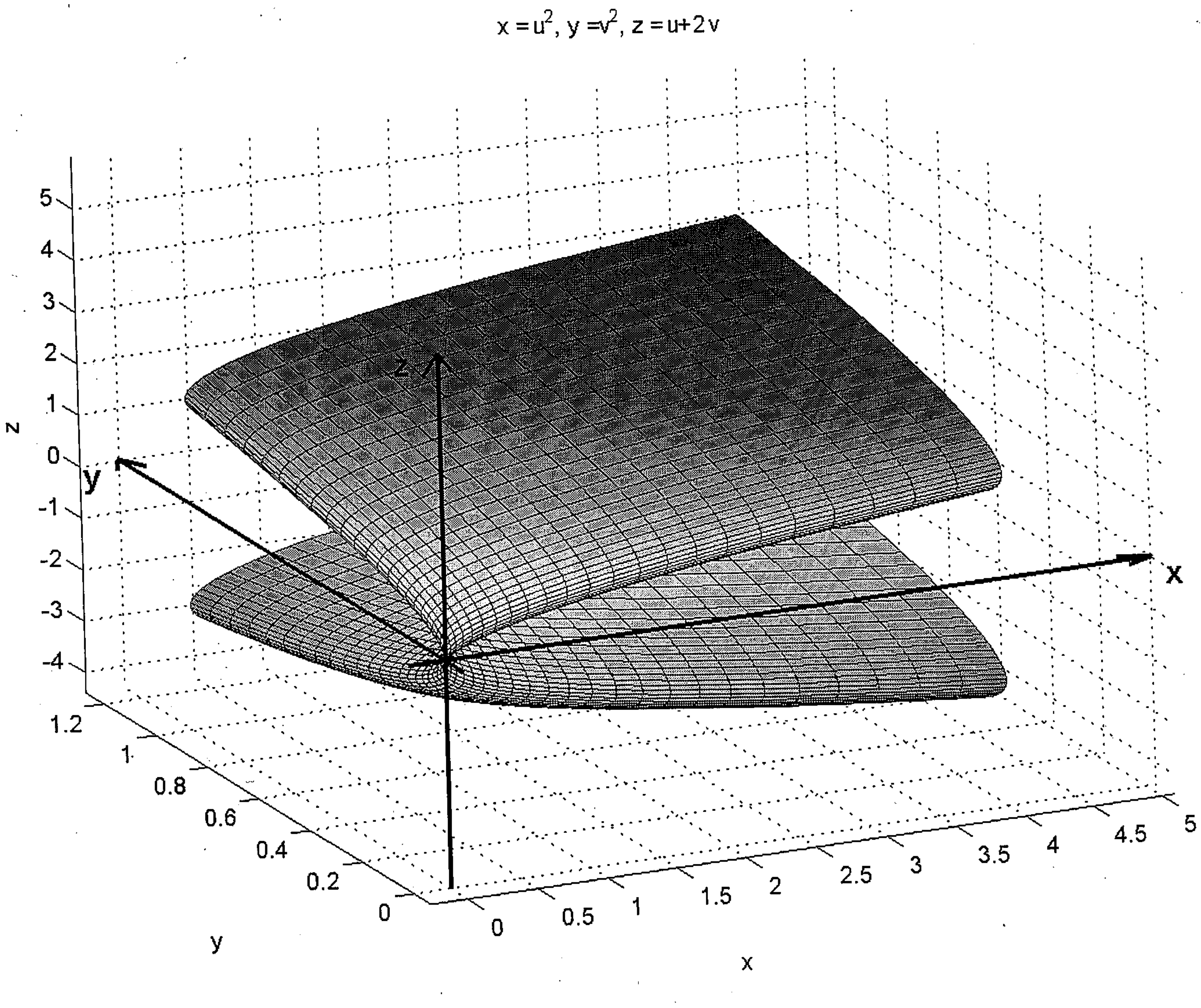
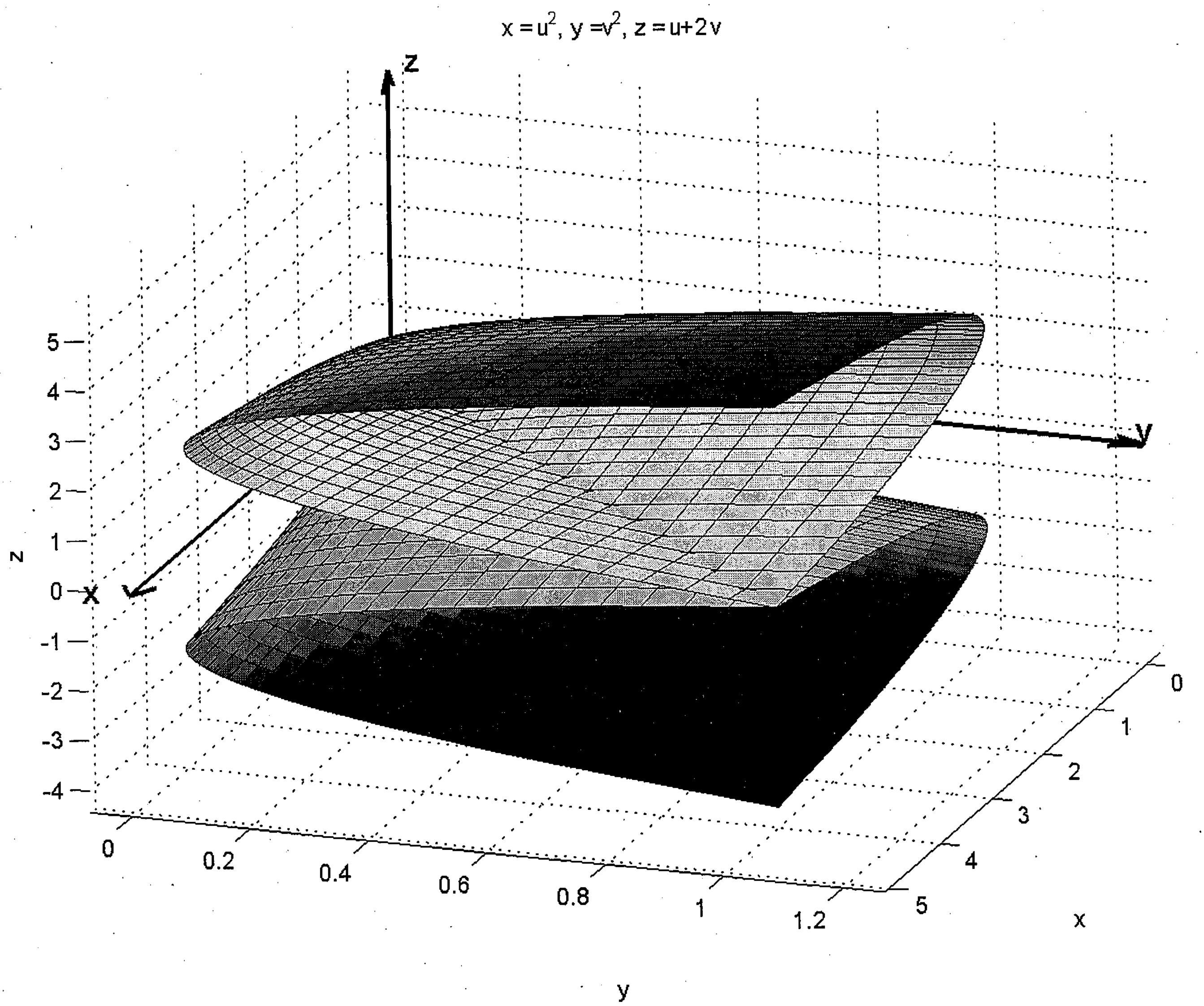
$\underline{r}_u \times \underline{r}_v \Big|_{(0,0)} = \underline{0} \Rightarrow$ surface is not smooth if $(u,v) = (0,0)$ lies inside $D$.

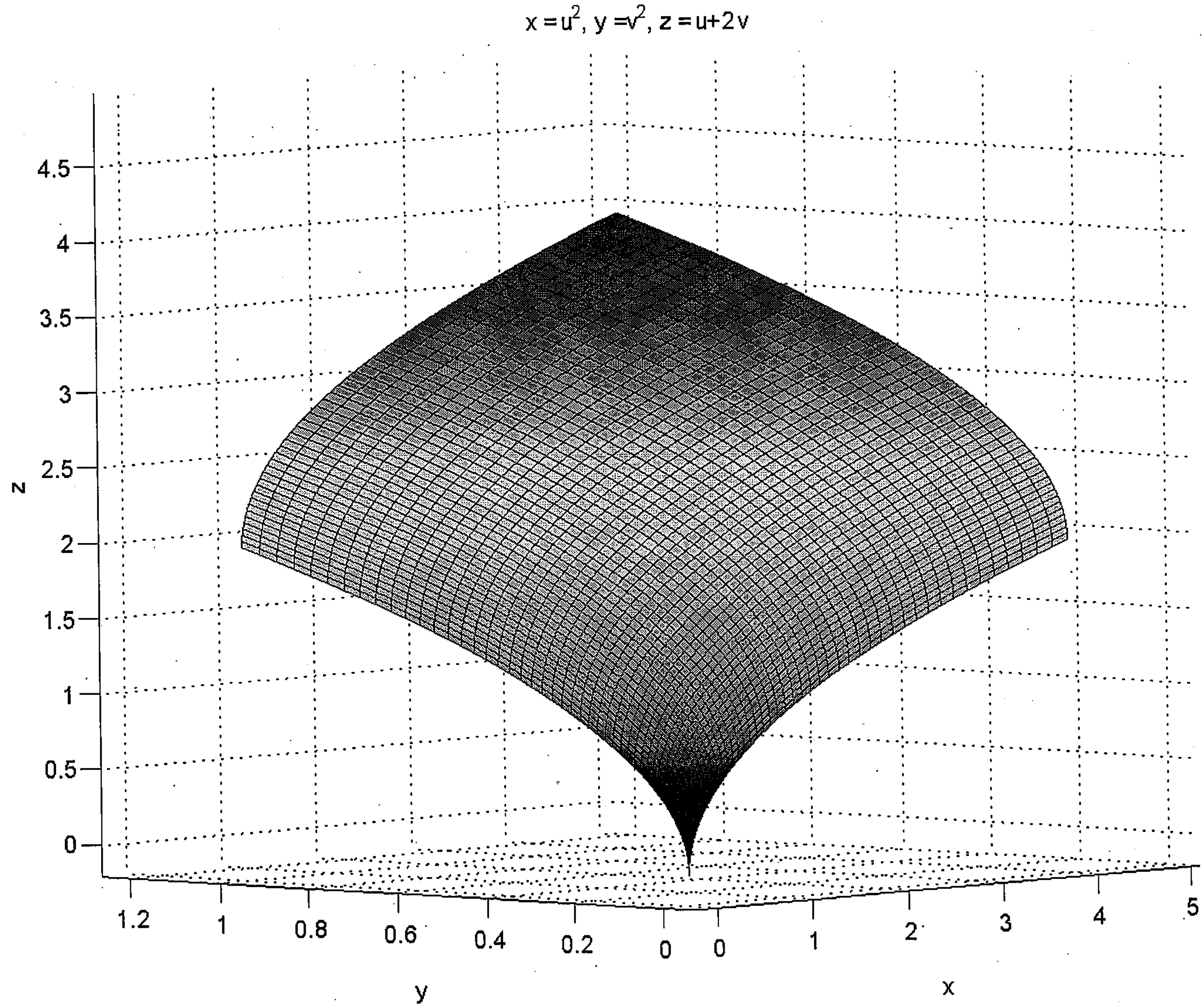
surface is "pinched" at $(0,0,0)$.



x = u^2, y = v^2, z = u+2v
5
4
3
2
1
0
-1
-2
-3
-4
z
y
x
1.2
1
0.8
0.6
0.4
0.2
0
y
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
x



x = u2, y =v2, z = u+2v
z
y
x
5
4
3
2
1
0
-1
-2
-3
-4
0
0.2
0.4
0.6
0.8
1
1.2
0
1
2
3
4
5



x =u^2, y =v^2, z =u+2v
z
y
x