24 Surface integrals

By the end of this section, you should be able to answer the following questions:

- What is a surface integral?
- How do you calculate the area of a parametric surface?
- How do you use surface integrals in applications such as calculating the mass of a "surface lamina" and finding the average temperature over a surface.

24.1 Area of a parametric surface

Let S be a smooth parametric surface given by

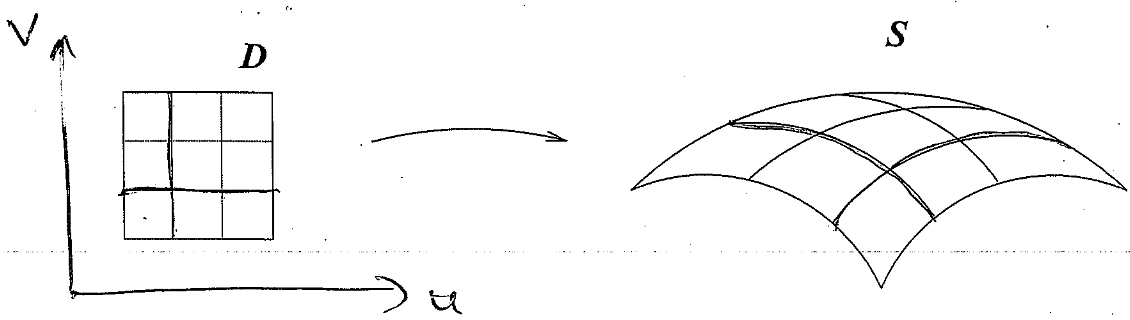
$$\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k},$$

where we assume for simplicity that the parameter domain is a rectangle in the u-v plane. To calculate the area of S, we work through the following steps:

1. Partition S into small patches.
2. Approximate each patch by a parallelogram lying in the tangent plane to the corner of the patch closest to the u-v origin.
3. Calculate the area ΔS of each parallelogram and add them to give an approximation to the area of S.
4. Take the limit as the dimensions of $\Delta S \to 0$ to obtain an exact expression for the area.

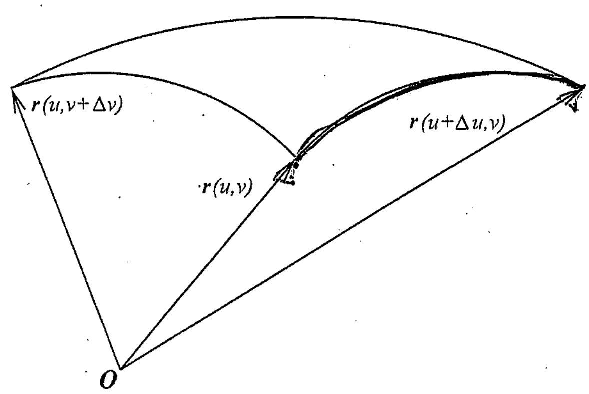
Let's have a closer look at each step.

1. A partition of S into patches will correspond to a partition of D (in the u-v plane) into small rectangles.



The dimensions of the rectangles in D will be $\Delta u \Delta v$.

2. Let one of the edges of a single patch be defined from parameter values (u, v) to $(u + \Delta u, v)$.

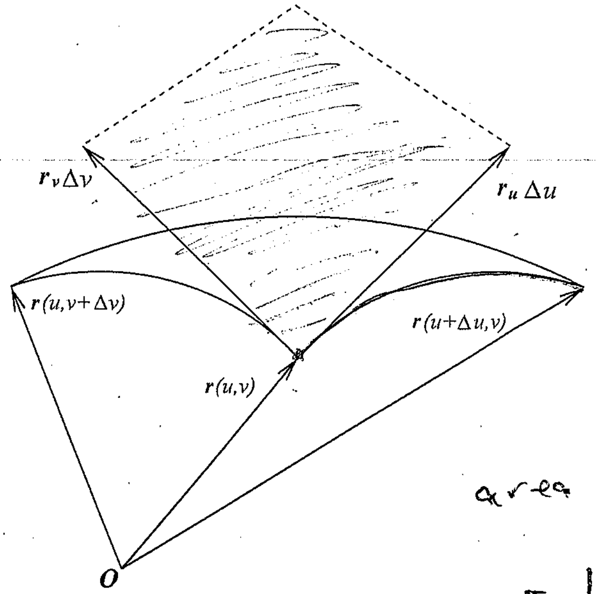


Using Pythagoras' law in three dimensions, we can approximate the length of this edge as

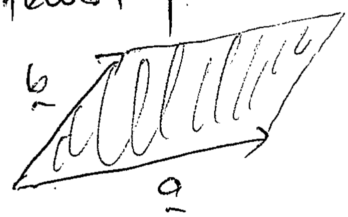
$$\begin{aligned} \text{length} &\approx \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \\ &= \left(\sqrt{\left(\frac{\Delta x}{\Delta u}\right)^2 + \left(\frac{\Delta y}{\Delta u}\right)^2 + \left(\frac{\Delta z}{\Delta u}\right)^2} \right) \Delta u \\ &\approx |\boldsymbol{r}_u| \Delta u, \end{aligned}$$

where in this case we have used $\Delta x = x(u+\Delta u, v) - x(u, v)$ etc (ie. the change is only in u). Similarly, for an edge of patch running from parameter values (u, v) to $(u, v + \Delta v)$ the length of that edge will be approximately $|\boldsymbol{r}_v| \Delta v$.

At the corner of the patch corresponding to parameter values (u, v), we can define the two vectors $\boldsymbol{r}_u \Delta u$ and $\boldsymbol{r}_v \Delta v$ which form two sides of a parallelogram, the side lengths of which coincide with our approximations to the lengths of the edges of the patch.



Stewart p825



area $= |a \times b|$

3. The vector $(\boldsymbol{r}_u \; \Delta u) \times (\boldsymbol{r}_v \; \Delta v)$ is normal to the surface (and hence the tangent plane) at that point. Its magnitude gives the area of the parallelogram we use to approximate the area of the patch ΔS. We then have

$$\Delta S \approx |\boldsymbol{r}_u \times \boldsymbol{r}_v| \; \Delta u \; \Delta v.$$

Adding these approximations for each patch in S gives us an approximation to the area of S:

$$\text{area of } S \approx \sum_i \Delta S_i = \sum_i |\boldsymbol{r}_{u_i} \times \boldsymbol{r}_{v_i}| \; \Delta u_i \; \Delta v_i.$$

4. Finally taking the limit as $\Delta u, \Delta v \to 0$ we obtain

$$\text{surface area } = \underbrace{\iint_S dS} = \iint_D |\boldsymbol{r}_u \times \boldsymbol{r}_v| \; du \; dv.$$

example of a "surface integral".

24.1.1 Application: find the surface area of the paraboloid $z = 1 - x^2 - y^2$ for $z \geq 0$.

p150: $\underset{\sim}{r}(r,\theta) = r\cos\theta\, \underset{\sim}{i} + r\sin\theta\, \underset{\sim}{j} + (1-r^2)\underset{\sim}{k}$

$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$ *

$\underset{\sim}{r}_r = \cos\theta\, \underset{\sim}{i} + \sin\theta\, \underset{\sim}{j} - 2r\underset{\sim}{k}.$

$\underset{\sim}{r}_\theta = -r\sin\theta\, \underset{\sim}{i} + r\cos\theta\, \underset{\sim}{j}$

$$\underset{\sim}{r}_r \times \underset{\sim}{r}_\theta = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= 2r^2\cos\theta\, \underset{\sim}{i} + 2r^2\sin\theta\, \underset{\sim}{j} + r\underset{\sim}{k}.$$

$$\Longrightarrow |\underset{\sim}{r}_r \times \underset{\sim}{r}_\theta| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= r\sqrt{4r^2+1}.$$

$$\text{Surface area} = \iint_S dS = \iint_D |\underset{\sim}{r}_r \times \underset{\sim}{r}_\theta|\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} r\sqrt{4r^2+1}\, d\theta\, dr$$

$$= \ldots = \frac{\pi}{6}(5\sqrt{5} - 1)$$

24.2 More on calculating surface integrals, applications

Let $f(x, y, z)$ be a scalar function in $\mathbb{R}^3$. We can ~~define~~ calculate the surface integral of f over a smooth parametric surface S in $\mathbb{R}^3$ as $S : \underline{r}(u,v)$

$$\lim_{\Delta S \to 0} \underbrace{\sum f(x^*, y^*, z^*)\Delta S}_{\text{(patches)}} \underset{\text{def.}}{=} \iint_S f(x, y, z)\, dS = \underset{\text{evaluate.}}{\iint_D} f(\boldsymbol{r}(u, v))|\boldsymbol{r}_u \times \boldsymbol{r}_v|\, du\, dv.$$

Surface integrals and double integrals have similar applications. Indeed, a double integral is merely a special case of a surface integral where the surface lies entirely in the x-y plane.

For example, if a thin sheet has the shape of a surface S and the mass density at the point (x, y, z) is $\rho(x, y, z)$, then the mass of the sheet is given by a surface integral:

$$\text{mass of sheet } = \iint_S \rho(x, y, z)\, dS.$$

Another application is in calculating the average value of a function over a surface. Let S be a smooth surface in $\mathbb{R}^3$. Then the average value of the function $f(x, y, z)$ over that surface is given by

$$\text{average value over surface } = \frac{1}{\text{area of } S} \iint_S f(x, y, z)\, dS.$$

If the surface S is a closed surface, it is convention to write

$$\oiint_S f(x, y, z)\, dS$$

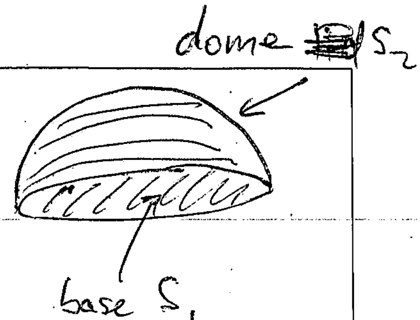
to represent the surface integral.

If S is a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only at their boundaries, then

$$\iint_S f(x, y, z)\, dS = \iint_{S_1} f(x, y, z)\, dS + \iint_{S_2} f(x, y, z)\, dS + \ldots + \iint_{S_n} f(x, y, z)\, dS.$$

Closed surfaces are often unions of smooth surfaces as demonstrated in the following example.

24.2.1 The function $T(x,y,z) = x^2 + y^2 + z^2 + 4$ gives the temperature at any point (x,y,z) on the surface of a solid hemisphere of radius 1 centred at the origin, defined for $z \geq 0$. Find the average temperature over the surface.



$$\text{avg. } T = \frac{\iint_S T\, dS}{\text{area of } S}$$

$$= \frac{\iint_{S_1} T\, dS + \iint_{S_2} T\, dS}{\iint_{S_1} dS + \iint_{S_2} dS}$$

$S = S_1 \cup S_2$

S_1 : disc in x-y plane

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$.

$\underline{r}(r,\theta) = r\cos\theta\, \underline{i} + r\sin\theta\, \underline{j}$ (polar coords!)

i.e. $\iint_{S_1} T\, dS = \iint_{S_1} (x^2+y^2+4)\, dS$ (since $z=0$)

Also $x^2+y^2=r^2$, check $\|\underline{r}_r \times \underline{r}_\theta\| = r$

(= Jacobian for polar coords!)

$$= \int_0^1 \int_0^{2\pi} (r^2+4) \cdot r\, d\theta\, dr$$

$$= \ldots = \frac{9}{2}\pi.$$

$\iint_{S_1} dS = \text{area of } S_1 = \pi \times 1^2 = \pi.$

S_2 : half sphere

$$\iint_{S_2} (\underbrace{x^2+y^2+z^2}_{\text{on } S_2,\ x^2+y^2+z^2=1}+4)\,dS = 5\iint_{S_2} dS$$

$$= 5\times(\text{area of } S_2)$$

$$= 5\times\tfrac{1}{2}\times 4\pi\times 1^2$$

$$= 10\pi.$$

$$\iint_{S_2} dS = 2\pi.$$

$\Rightarrow$ avg. of $T(x,y,z) = x^2+y^2+z^2+4$ on S is

$$\overline{T} = \frac{\frac{9}{2}\pi + 10\pi}{\pi + 2\pi} = \frac{29}{6}.$$

Steps usually

① parameterize S with $\underline{r}(u,v)$, $(u,v)\in D$.

② Calculate $|\underline{r}_u \times \underline{r}_v|$

③ $$\iint_S f(x,y,z)\,dS = \iint_D f(\underline{r}(u,v))\,|\underline{r}_u\times\underline{r}_v|\,du\,dv$$