25 Flux integrals and Gauss' divergence theorem

By the end of this section, you should be able to answer the following questions:

- What is a flux integral?
- How do you use a flux integral to calculate the flux of a vector field across a surface?
- What is Gauss' divergence theorem and under what conditions can it be applied?
- How do you apply Gauss' divergence theorem?

We have already been introduced to the idea of flux of a variable vector field across a curve (in $\mathbb{R}^2$) and the flux of a constant vector field across rectangular surfaces (in $\mathbb{R}^3$). In this section we look at calculating the flux across smoothly parametric surfaces.

25.1 Orientable surfaces

Let S be a smooth surface. If we can choose a unit vector that is normal to S at every point so that $\boldsymbol{n}$ varies continuously over S, we call S an *orientable* surface. The choice of $\boldsymbol{n}$ provides S with an *orientation*. There are only ever two possible orientations.

An example of an orientable surface is the surface of a sphere. The two possible orientations are out of the sphere or into the sphere.

An example of a non-orientable surface is a Möbius strip (see Stewart page 1121).

The orientation of a surface is important when considering flux through that surface. The orientation we choose is always the direction of positive flux.

25.2 The flux integral

For a vector field $\boldsymbol{v}(x, y, z)$, we are interested in the flux of $\boldsymbol{v}$ across a smooth orientable parametric surface S in $\mathbb{R}^3$, parametrised by $\boldsymbol{r}(u, v)$, with u and v defined over some domain D. Let $\boldsymbol{n}(u, v)$ be a unit vector normal to the surface S which defines the orientation of the surface (and hence the direction of positive flux).

It would be most convenient to consider the context of fluid flow with $\boldsymbol{v}(x, y, z)$ being the velocity of a fluid at the point (x, y, z).

To calculate the flux through S, we work through the following steps:

1. Partition S into small patches.
2. Approximate each patch by a parallelogram lying in the tangent plane to the corner of the patch closest to the u-v origin.
3. Approximate the flux through each parallelogram of approximate area ΔS and add them to give an approximation to the total flux through S.
4. Take the limit as the dimensions of $\Delta S \to 0$ to obtain an exact expression for the flux.

Let's have a closer look at these steps.

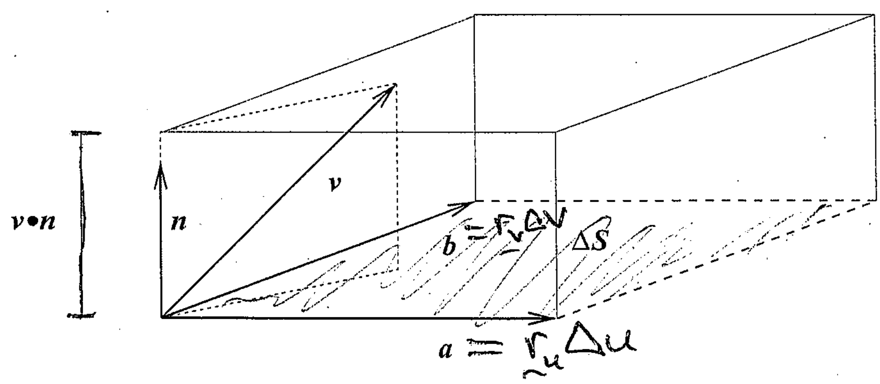
1,2. Steps 1 and 2 are exactly the same as steps 1 and 2 on page 156-157 of our calculation of surface area.

3. We approximate the flux through one patch by treating $\boldsymbol{v}$ as constant over the patch (ie. the patch is small enough for this to be a decent approximation). Since we have already approximated the shape of the patch as a parallelogram, we need to work out the flux of a constant vector through a parallelogram.

   To this end, consider the parallelogram defined by the two (non-parallel) vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. If we take the area of the patch to be ΔS, it can be seen from the diagram below that the flux (volume per unit time if $\boldsymbol{v}$ is velocity) passing through the parallelogram is

$$\text{flux across parallelogram } \approx \boldsymbol{v} \cdot \boldsymbol{n}\, \Delta S.$$

(see p131)



We can take $\boldsymbol{n} = \dfrac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|}$, and the area of the parallelogram is $|\boldsymbol{a} \times \boldsymbol{b}| \approx \Delta S$. We then have

$$\text{flux across parallelogram } \approx \left(\boldsymbol{v} \cdot \frac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|}\right)|\boldsymbol{a} \times \boldsymbol{b}| = \boldsymbol{v} \cdot (\boldsymbol{a} \times \boldsymbol{b}).$$

Scalar triple product

(vol. of parallelopiped p827 Stewart)

$$\underline{v}\cdot\underline{n}\,\Delta S = \frac{\underline{v}\cdot(\underline{r}_u\times\underline{r}_v)\Delta u\Delta v}{\cancel{|\underline{r}_u\times\underline{r}_v|\Delta u\Delta v}} \qquad \cancel{|\underline{r}_u\times\underline{r}_v|\Delta u\Delta v}$$

As shown previously, a patch of surface can be approximated by a parallelogram determined by the two vectors $\boldsymbol{r}_u\Delta u$ and $\boldsymbol{r}_v\Delta v$. Hence we have

$$\text{flux across one patch } \approx \boldsymbol{v}\cdot\boldsymbol{n}\;\Delta S = \boldsymbol{v}\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\Delta u\Delta v. \quad *$$

* Note that we need to check that the vector $\boldsymbol{r}_u \times \boldsymbol{r}_v$ points in the direction of positive flux. If not, we use $\boldsymbol{r}_v \times \boldsymbol{r}_u$. *

Adding these approximations over the entire surface S, we obtain

$$\text{flux across } S \approx \sum_i \boldsymbol{v}_i\cdot\boldsymbol{n}_i\;\Delta S_i = \sum_i \boldsymbol{v}(u_i,v_i)\cdot(\boldsymbol{r}_{u_i}\times\boldsymbol{r}_{v_i})\Delta u_i\Delta v_i.$$

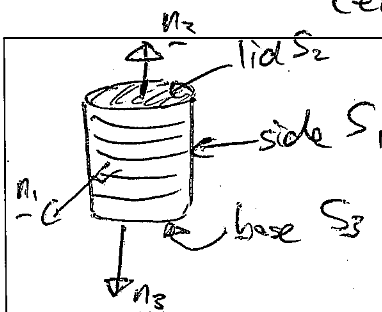
4. To obtain an exact expression for the flux across S we take the limit as $\Delta u, \Delta v \to 0$.

$$\text{flux across } S = \iint_S \boldsymbol{v}\cdot\boldsymbol{n}\;dS = \iint_D \boldsymbol{v}\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\;du\,dv.$$

This expression is called a *flux integral* and is used to calculate the flux of any vector field across a smooth orientable surface, not just fluids with a given velocity field.

direction positive flux.

25.2.1 **Calculate the net outward flux of $\boldsymbol{F}(x,y,z) = z\boldsymbol{i} + y\boldsymbol{j} + x\boldsymbol{k}$ across the surface of the cylindrical solid given by** $\{(x,y,z) \mid x^2+y^2 \leq 1,\ 0 \leq z \leq 2\}$. centre z-axis, radius 1.



net outward flux

$$= \oint\!\!\!\oint_S \underline{F}\cdot\underline{n}\,dS$$

$$= \iint_{S_1} \underline{F}\cdot\underline{n}_1\,dS + \iint_{S_2} \underline{F}\cdot\underline{n}_2\,dS + \iint_{S_3} \underline{F}\cdot\underline{n}_3\,dS.$$

S_1: $\underline{r}(\theta,z) = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} + z\underline{k}$ (cyl. rad 1)

$0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 2.$

$\underline{r}_z = \underline{k}$, $\underline{r}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$

$$\underline{r}_z \times \underline{r}_\theta = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = -\cos\theta\,\underline{i} - \sin\theta\,\underline{j}$$

Which is directed <u>into</u> the cylinder.

$\Rightarrow \underline{r}_\theta \times \underline{r}_z = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$ is directed out.

$\underline{F}(\underline{r}(\theta,z)) = z\,\underline{i} + \sin\theta\,\underline{j} + \cos\theta\,\underline{k}$.

$$\Rightarrow \iint_{S_1} \underline{F}\cdot\underline{n}_1\,dS = \iint_D \left[\underline{F}(\underline{r}(\theta,z))\cdot(\underline{r}_\theta \times \underline{r}_z)\right] d\theta\,dz$$

$$= \int_0^2 \int_0^{2\pi} (\cancel{z\cos\theta} + \underline{\sin^2\theta})\,d\theta\,dz$$

$$= \left(\int_0^2 dz\right)\left(\int_0^{2\pi}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)\right)d\theta\right) \quad \text{(trig. id.)}$$

$$= \dots = 2\pi.$$

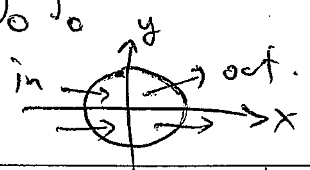
S_2: disc in plane $\boxed{z=2}$, $\underline{n}_2 = \underline{k}$.

$\underline{r}(r,\theta) = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + 2\underline{k}$.

$\underline{F}\cdot\underline{n}_2 = x.$, check $|\underline{r}_r \times \underline{r}_\theta| = r$

$$\Rightarrow \iint_{S_2} \underline{F}\cdot\underline{n}_2\,dS = \iint_{S_2} x\,dS \quad \text{(regular surface integral)}$$

$$= \int_0^1 \int_0^{2\pi} r\cos\theta \cdot r\,d\theta\,dr = 0$$



S_3: $\underline{n}_3 = -\underline{k} \Rightarrow$ out flux $= 0$ (similar to S_2 case)

$\Rightarrow$ net outward flux $= 2\pi + 0 + 0$

25.3 Gauss' divergence theorem

On page 142 we saw the flux form of Green's theorem:

$$\oint_C \boldsymbol{v}(x,y)\cdot\boldsymbol{n}\; dS = \iint_D \operatorname{div}(\boldsymbol{v}(x,y))\; dA.$$

The left hand side is essentially a flux integral in two dimensions, with $\boldsymbol{n}$ being an outwardly pointing unit normal vector to the curve C. The right hand side was derived from our realisation of the divergence as the "flux density".

It would be natural to ask if it is possible to extend this result to three dimensions.

Given a vector field in three dimensions, $\boldsymbol{F}(x,y,z)$, we have seen that the net outward flux across a closed, smooth, orientable surface S is given by $\oiint_S \boldsymbol{F}\cdot\boldsymbol{n}\; dS$, where $\boldsymbol{n}$ is an outwardly pointing unit normal.

We have also seen that its divergence (div$\boldsymbol{F}$) can be viewed as the flux density, so

$$\operatorname{div}\boldsymbol{F} = \lim_{\Delta V\to 0} \frac{\text{flux of } \boldsymbol{F} \text{ out of } \Delta V}{\Delta V}.$$

Hence we expect to be able to calculate the net outward flux across a closed, smooth, orientable surface S as the triple integral of the flux density (ie. div$\boldsymbol{F}$) over the volume enclosed by S.

Indeed, this is true, with $\boldsymbol{F}$ and S subject to certain conditions. The result is known as *Gauss' divergence theorem*:

Let S be a piecewise smooth, orientable, closed surface enclosing a region V in $\mathbb{R}^3$. Let $\boldsymbol{F}(x,y,z)$ be a vector field whose component functions are continuous and have continuous partial derivatives in V. Then

$$\oiint_S \boldsymbol{F}\cdot\boldsymbol{n}\; dS = \iiint_V \operatorname{div}(\boldsymbol{F})dV,$$

where $\boldsymbol{n}$ is the outwardly directed unit normal to S.

This theorem connects the flux of a vector field out of a volume with the flux through its surface. It says that we can calculate the net outward flux either as a closed surface integral, or as a triple integral.

25.3.1 Use Gauss' divergence theorem to calculate the net outward flux of $F(x, y, z) = zi + yj + xk$ across the surface of the cylindrical solid given by $\{(x, y, z) \mid x^2 + y^2 \le 1,\ 0 \le z \le 2\}$.

Same prob. as p164–165.

use $\oiint_S \underline{F} \cdot \underline{n} \, dS = \iiint_V \underline{\nabla} \cdot \underline{F} \, dV$

$$\underline{\nabla} \cdot \underline{F} = \cancel{\frac{\partial}{\partial x}(z)} + \frac{\partial}{\partial y}(y) + \cancel{\frac{\partial}{\partial z}(x)}$$

$$= 1.$$

Gauss $\Rightarrow \oiint_S \underline{F} \cdot \underline{n} \, dS = \iiint_V 1 \, dV$

$= \text{vol. of } V$

$= \pi \times 1^2 \times 2 = 2\pi.$