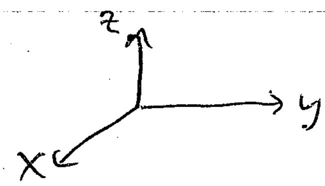
26 Curl of a vector field

By the end of this section, you should be able to answer the following questions:

- How do you calculate the curl of a given vector field?
- What is the significance of curl?
- How do you test whether or not a given three dimensional vector field is conservative?



26.1 Calculating curl

If (x, y, z) is a right handed Cartesian coordinate system and $\boldsymbol{v}(x,y,z) = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ is a differentiable vector field, then define

$$\mathrm{curl}(\boldsymbol{v}) = \nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\boldsymbol{k}. \quad \# \mathrm{def}.$$

Note that $\mathrm{curl}(\boldsymbol{v})$ is a vector field. $\underline{\nabla} = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$.

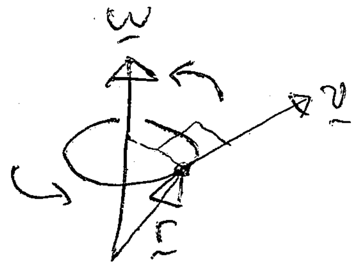
26.1.1 Example: let $\boldsymbol{v} = yz^2\boldsymbol{i} + zx^2\boldsymbol{j} + xy^2\boldsymbol{k}$. Find $\mathrm{curl}(\boldsymbol{v})$.

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & zx^2 & xy^2 \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial}{\partial y}(xy^2) - \frac{\partial}{\partial z}(zx^2)\right) - \underline{j}\left(\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial z}(yz^2)\right) + \underline{k}\left(\frac{\partial}{\partial x}(zx^2) - \frac{\partial}{\partial y}(yz^2)\right)$$

$$= (2xy - x^2)\underline{i} - (y^2 - 2yz)\underline{j} + (2zx - z^2)\underline{k}.$$

26.2 Understanding curl



For the rotation of a rigid body about a fixed axis with angular velocity $\boldsymbol{w}$, the velocity at a point P, whose position vector is $\boldsymbol{r}$, is given by $\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{r}$.

If we choose the axis of rotation to be the z-axis, then $\boldsymbol{w} = \omega\boldsymbol{k}$. Calculate curl($\boldsymbol{v}$).

$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, $\underline{w} = \omega\underline{k}$.

$$\underline{v} = \underline{w} \times \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix}$$

$$= -\omega y\,\underline{i} + \omega x\,\underline{j}$$

The curl of $\underline{v}$ is then

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= \underline{i}(0) - \underline{j}(0) + \underline{k}(\omega + \omega) = 2\omega\underline{k}$$

$\Rightarrow$ curl ($\underline{v}$) is proportional $= 2\underline{w}$
to the angular velocity.

go to page 145.
"paddle wheel experiment".

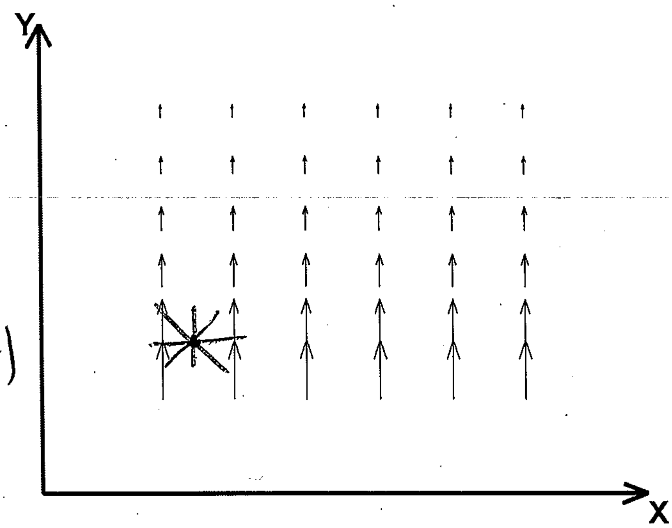
In general, curl($\boldsymbol{v}$) characterises the rotation of a vector field. We will investigate this further in the next section.

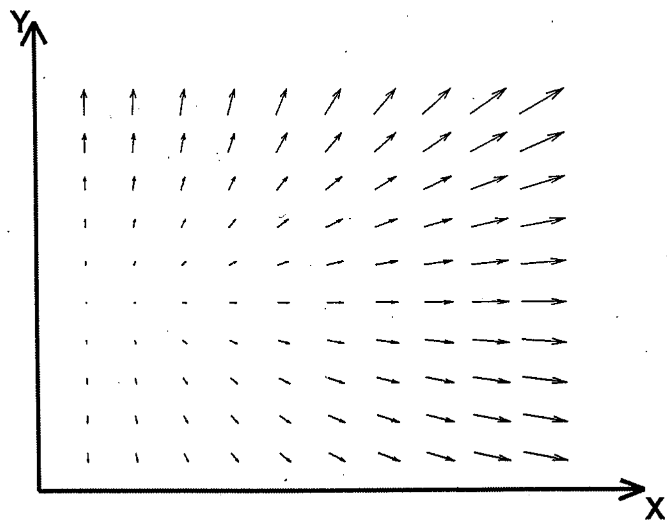
22.4.2 For the following graphs of vector fields, determine whether the divergence is positive, negative or zero.

from p171

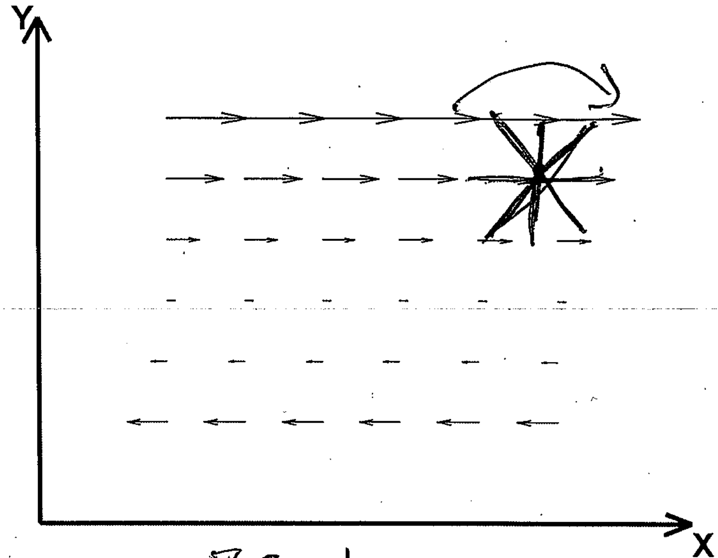
estimate...

no rotation.
of wheel
$\Rightarrow$ curl $= \underline{0}$
($\underline{k}$ component)

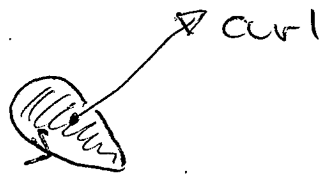




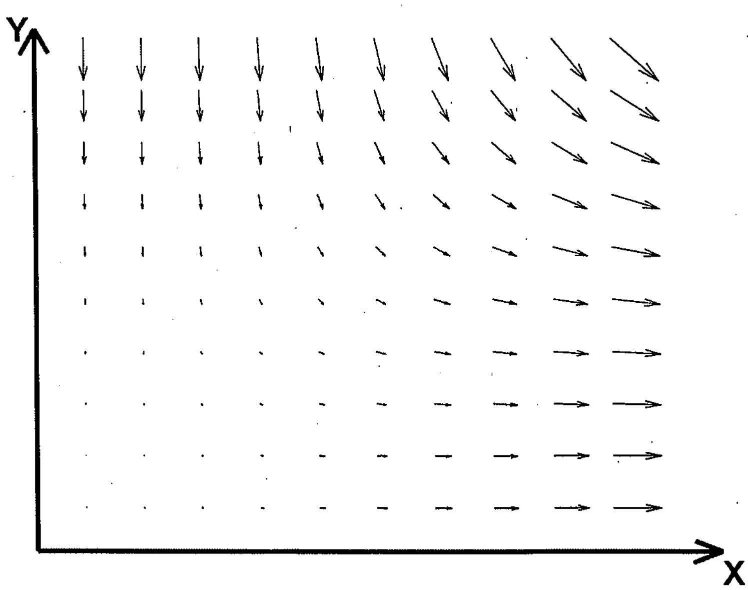
estimate...
clockwise
rotation
curl $= a\underline{k}$
where $a<0$
curl directed
into page



In general



"idea of divergence
and curl"..



26.3 Conservative fields revisited

It turns out that the curl of a vector field is exactly what we need to generalise the result at the bottom of page 126 to three dimensions.

Show that if $\boldsymbol{F}$ is a conservative vector field, then curl$\boldsymbol{F} = \mathbf{0}$.

$$\underline{F} = \underline{\nabla} f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}.$$

$$\operatorname{curl}(\underline{\nabla} f) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - \underline{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \underline{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = \underline{0}$$

by Clairaut's theorem (assuming cts derivatives).

$$\operatorname{curl}(\operatorname{grad} f) = \underline{0}.$$

Indeed, the diagram on page 126 that outlines our logic can be extended directly to the three dimensional case. The only difference is the condition which will serve as our test for conservative fields, namely curl$\boldsymbol{F} = \mathbf{0}$.

The proofs of the links in the diagram for the three dimensional case below are very similar to those used in the two dimensional case. The only detail that is significantly different is showing that if $\text{curl}\boldsymbol{F} = \boldsymbol{0}$ then $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$. Note also that $\boldsymbol{F}$ must be a vector field defined everywhere in $\mathbb{R}^3$ with continuous partial derivatives. The proof of that part of the diagram requires a generalisation of Green's theorem known as *Stokes' theorem*, which we will investigate in the next section.

$$\begin{tikzcd}
 & \boldsymbol{F}\text{ conservative} \arrow[dl, shift left] \arrow[dr, "\text{Clairaut's theorem.}"] & \\
\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}\text{ path independent} \arrow[ur, shift left] \arrow[dr, shift left] & & \text{curl}\boldsymbol{F} = 0 \arrow[dl, "\text{* Stokes' thm (see next chapter).}"] \\
 & \displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0,\ \forall\text{ closed } C \arrow[ul, shift left] &
\end{tikzcd}$$

The main consequence of this diagram is that we have the following test for a conservative vector field in three dimensions:

$$\boxed{\text{A vector field } \boldsymbol{F} \text{ is conservative if and only if } \text{curl}\boldsymbol{F} = \boldsymbol{0}.}$$

($\underline{F}$ defined everywhere with cts partial derivatives)

26.3.1 **Determine whether or not the vector field $\boldsymbol{F} = (1 + yz)\boldsymbol{i} + (1 + xz)\boldsymbol{j} + xy\boldsymbol{k}$ is conservative.**

$$\text{Curl}(\underline{F}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1+yz & 1+xz & xy \end{vmatrix}$$

$$= \underline{i}(x - x) - \underline{j}(y - y) + \underline{k}(z - z) = \underline{0}$$

$\Rightarrow \underline{F}$ is conservative.