27 Stokes' theorem

By the end of this section, you should be able to answer the following questions:

- What is Stokes' theorem and under what conditions can it be applied?
- How do you apply Stokes' theorem?
- What is the circulation of a vector field?

27.1 Summary of surfaces and curves

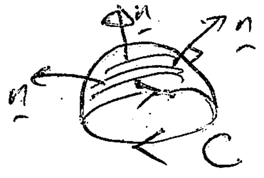
Here we summarise the different types of curves and surfaces which we need to understand Stokes' theorem. Although most of these definitions have already been given, you may find it useful to have all of this information in one place so you can review at a glance.

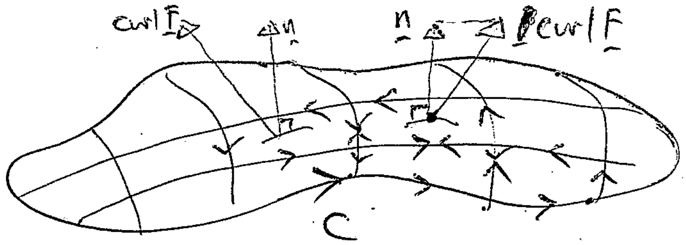
27.1.1 Surfaces

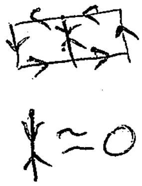
- *Smooth*: the surface normal vector depends continuously on the points on the surface.
- *Piecewise smooth*: the surface consists of finitely many smooth surfaces intersecting only at their boundaries.
- *Oriented* (or *orientable*): the direction of the positive normal vector can be continued uniquely and continuously across the whole surface (especially if the surface is piecewise smooth).

27.1.2 Curves

- *Smooth*: the tangent at each point on the curve is unique and varies continuously.
- *Piecewise smooth*: the curve consists of finitely many smooth curves.
- *Simple*: the curve never intersects itself anywhere between its endpoints.







27.2 Stokes' theorem

> Let S be a piecewise smooth, orientable surface in $\mathbb{R}^3$ and let the boundary of S be a piecewise smooth, simple, closed curve C. Let $\boldsymbol{F}(x, y, z)$ be a continuous vector function with continuous first partial derivatives in some domain containing S. Then
>
> $$\iint\limits_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n}\, dA = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r},$$
>
> where $\boldsymbol{n}$ is a unit normal vector of S, and the integration around C is taken in the direction using the "right hand rule" with $\boldsymbol{n}$.

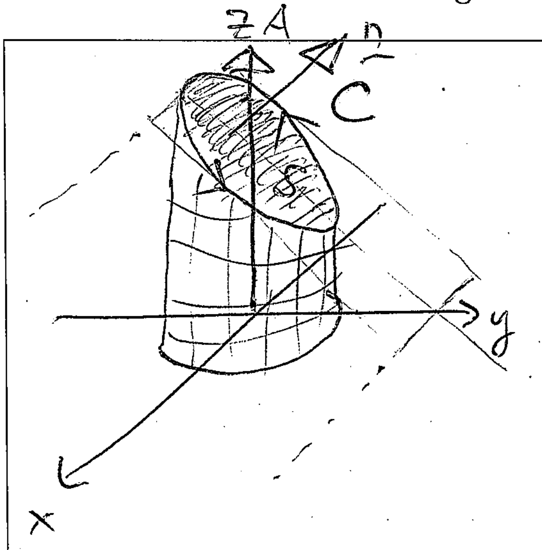
special case proof in Stewart p1129–1130, for S given by $z = g(x,y)$

27.2.1 Relation to Green's theorem

Recall Green's theorem in the plane. It relates a line integral on a boundary to a double integral over a region in the plane. Roughly speaking, Stokes' theorem is a 3-D version of this: it relates a surface integral on a piece of surface (in 3-D) to a line integral on the boundary of the surface.

In fact, note that if the surface is in the x-y plane with $\boldsymbol{n} = \boldsymbol{k}$, Stokes' theorem reduces to Green's theorem, since the $\boldsymbol{k}$ component of curl$\boldsymbol{F}$ is just $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$.

27.2.2 Verify Stokes' theorem where C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise when looking from above, and $\boldsymbol{F} = [-y^2, x, z^2]$.



Calculate $\oint_C \underline{F} \cdot d\underline{r}$,

$\iint_S (\text{curl}\, \underline{F}) \cdot \underline{n}\; dS$

$$\int_C \left(\underline{F} \cdot \frac{dr}{dt}\right) dt.$$

$\oint_C$: parameterize C, $\{x^2+y^2=1,\ y+z=2\}$.

$\underline{r}(t) = \cos t\,\underline{i} + \sin t\,\underline{j} + (2-\sin t)\underline{k}$

$0 \le t \le 2\pi$.

$\dfrac{d\underline{r}}{dt} = -\sin t\,\underline{i} + \cos t\,\underline{j} - \cos t\,\underline{k}$.

$\underline{F}(\underline{r}(t)) \cdot \dfrac{d\underline{r}}{dt} = (-\sin^2 t, \cos t, 4-4\sin t+\sin^2 t)\cdot(-\sin t, \cos t, -\cos t)$

$\Rightarrow \oint_C \underline{F}\cdot d\underline{r} = \int_0^{2\pi} (\sin^3 t + \cos^2 t - 4\cos t + 4\cos t\sin t - \cos t\sin^2 t)\,dt$

$\boxed{* \text{ If } m \text{ or } n \text{ is } \underline{\text{odd}} \text{ then } \int_0^{2\pi} \sin^m t\cos^n t\,dt = 0 *}$

$= \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi}\left(\tfrac{1}{2} + \tfrac{1}{2}\underbrace{\cos 2t}_{0}\right)dt = \tfrac{1}{2}\times 2\pi = \pi$

$$\underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)\underline{k}.$$

Take S to be the disc (ellipse) in the plane $y+z=2$.

S: $\underline{r}(r,\theta) = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + (2-r\sin\theta)\underline{k}$.

$0 \le r \le 1$

$0 \le \theta \le 2\pi$.

$\underline{r}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} - \sin\theta\,\underline{k}$.

$\underline{r}_\theta = -r\sin\theta\,\underline{i} + r\cos\theta\,\underline{j} - r\cos\theta\,\underline{k}$

$$\Rightarrow \underline{r}_r\times\underline{r}_\theta = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & -\sin\theta \\ -r\sin\theta & r\cos\theta & -r\cos\theta \end{vmatrix} = \dots = r\underline{j} + r\underline{k}.$$

(direction okay)

$(\mathrm{curl}\,\underline{F}(r,\theta))\cdot(\underline{r}_r\times\underline{r}_\theta) = (1+2r\sin\theta)r$

$$\Rightarrow \iint_S (\mathrm{curl}\,\underline{F})\cdot\underline{n}\,dS = \int_0^1\int_0^{2\pi}(1+2r\underbrace{\sin\theta}_{\to 0})r\,d\theta\,dr$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r\,dr\right) = 2\pi\times\tfrac{1}{2} = \pi.$$

27.3 Circulation

Let $\boldsymbol{v}$ represent the velocity field of a fluid and C is a piecewise smooth, simple, closed curve. We have

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = \oint_C \boldsymbol{v} \cdot \boldsymbol{T}\, dS,$$

where $\boldsymbol{T}$ is a unit tangent vector in the direction of the orientation of the curve. The dot product $\boldsymbol{v} \cdot \boldsymbol{T}$ is the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{T}$ (and hence the curve), so we can interpret $\oint_C \boldsymbol{v} \cdot \boldsymbol{T}\, dS$ as a measure of the tendency of the fluid to move around the curve C. We call this quantity the *circulation* of $\boldsymbol{v}$ around C.

Now define a small circle C_a of radius a about a point P_0, such that the disc S_a enclosed by C_a is normal to the vector $\boldsymbol{n}(P_0)$. Our aim here is to better understand curl$\boldsymbol{v}$.

Since curl$\boldsymbol{v}$ is continuous, we approximate curl$\boldsymbol{v}$ over S_a as curl$\boldsymbol{v}(P_0)$. Stokes theorem then gives us

$$\begin{aligned}
\oint_{C_a} \boldsymbol{v} \cdot d\boldsymbol{r} &= \iint_{S_a} \operatorname{curl}\boldsymbol{v} \cdot \boldsymbol{n}\, dS \\
&\approx \iint_{S_a} \operatorname{curl}\boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0)\, dS \\
&= \operatorname{curl}\boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0) \iint_{S_a} dS \\
&= \operatorname{curl}\boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0)(\pi a^2) \\
\Rightarrow \quad \operatorname{curl}\boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0) &\approx \frac{1}{\pi a^2} \oint_{C_a} \boldsymbol{v} \cdot d\boldsymbol{r} \\
&\approx \frac{\text{circulation around disc}}{\text{area of disc}}.
\end{aligned}$$

This approximation improves as $a \to 0$. Indeed

$$\operatorname{curl}\boldsymbol{v}(P_0) \cdot \boldsymbol{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \oint_{C_a} \boldsymbol{v} \cdot d\boldsymbol{r}.$$

Note that this has a maximum value when curl$\boldsymbol{v}(P_0)$ and $\boldsymbol{n}(P_0)$ have the same direction.

In particular, if we take $\boldsymbol{n}(P_0)$ to be each of the coordinate unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, we have the following: The $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ components of curl$\boldsymbol{v}(P_0)$ give the *circulation density* at P_0 in planes normal to each of the $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ respectively. The magnitude of curl$\boldsymbol{v}(P_0)$ gives the maximum circulation density about P_0 in a plane normal to curl$\boldsymbol{v}(P_0)$.

27.4 Curl fields and vector potentials

One immediate consequence is that if there are two different surfaces S_1 and S_2 satisfying the criteria of Stokes' theorem, both with the same boundary curve C, then

$$\iint_{S_1} \text{curl}\boldsymbol{F} \cdot \boldsymbol{n}_1 \, dS = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{S_2} \text{curl}\boldsymbol{F} \cdot \boldsymbol{n}_2 \, dS.$$

We have that if S is a closed surface satisfying all of the other criteria of Stokes' theorem, and if we define C to be any closed curve lying on S, so that S_1 and S_2 are two open surfaces whose union makes up S and whose common boundary is C, then

$$\begin{aligned}
\oiint_S \text{curl}\boldsymbol{F} \cdot \boldsymbol{n} \, dS &= \iint_{S_1} \text{curl}\boldsymbol{F} \cdot \boldsymbol{n}_1 \, dS + \iint_{S_2} \text{curl}\boldsymbol{F} \cdot \boldsymbol{n}_2 \, dS \\
&= \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} \\
&= \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} - \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0;
\end{aligned}$$

since the orientation of C as a boundary to S_1 will be in the opposite direction to that of S_2.

Let $\boldsymbol{F}$ be a vector field satisfying $\boldsymbol{F} = \text{curl}\boldsymbol{G}$ for some vector field $\boldsymbol{G}$. We call $\boldsymbol{F}$ a *curl field* and $\boldsymbol{G}$ a corresponding *vector potential.*

The above result says that the net outward flux of a curl field across any closed surface is zero.

We can verify that $\text{div}(\text{curl}\boldsymbol{G}) = 0$ for any vector field $\boldsymbol{G}$. Consequently we should not be too surprised by the above result, since Gauss' divergence theorem says that

$$\oiint_S (\text{curl}\boldsymbol{G}) \cdot \boldsymbol{n} \, dS = \iiint_V \text{div}(\text{curl}\boldsymbol{G}) \, dV = 0.$$

In fact, it turns out that we have the following test for curl fields:

Let $\boldsymbol{F}$ be a vector field whose components and their partial derivatives are continuous. If every closed surface in the domain of $\boldsymbol{F}$ only encloses points which are also in the domain of $\boldsymbol{F}$, and if $\text{div}\boldsymbol{F} = 0$, then there exists some $\boldsymbol{G}$ such that $\boldsymbol{F} = \text{curl}\boldsymbol{G}$. That is, $\boldsymbol{F}$ is a curl field.