28 Gaussian elimination and linear equations

By the end of this section, you should be able to answer the following questions:

- How do you use Gaussian elimination to find the row echelon form of a matrix?
- What are the three cases for solutions to systems of linear equations?
- How do you solve a system of linear equations?
- What are elementary matrices and how do they relate to elementary row operations?

Say we have m linear equations in n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

We can write these equations in matrix form: $A\boldsymbol{x} = \boldsymbol{b}$.

$A = [a_{ij}]$ is the $m \times n$ coefficient matrix.

$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is the column vector of unknowns, and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ is the column vector of the right hand side.

Note: a_{ij}, $b_j \in \mathbb{R}$ or $\mathbb{C}$.

28.1 Gaussian Elimination

To solve $A\boldsymbol{x} = \boldsymbol{b}$:
write *augmented matrix*: $[A|\boldsymbol{b}]$.

1. Find the left-most non-zero column, say column j.
2. Interchange top row with another row if necessary, so top element of column j is non-zero. (The **pivot**.)
3. Subtract multiples of row 1 from all other rows so all entries in column j below the top are then 0.

4. Cover top row; repeat **1** above on rest of rows.
Continue until all rows are covered, or until only $00\ldots0$ rows remain.

The result is a triangular system, easily solved by *back substitution*: solve the last equation first, then 2nd last equation and so on.

28.1.1 Example

Use Gaussian elimination to solve:

$$\begin{aligned} x_3 - x_4 &= 2 \\ -9x_1 - 2x_2 + 6x_3 - 12x_4 &= -7 \\ 3x_1 + x_2 - 2x_3 + 4x_4 &= 2 \\ 2x_3 &= 6 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 2 \\ -9 & -2 & 6 & -12 & -7 \\ 3 & 1 & -2 & 4 & 2 \\ 0 & 0 & 2 & 0 & 6 \end{array}\right) \quad R_1 \leftrightarrow R_3$$

$$\left(\begin{array}{cccc|c} 3 & 1 & -2 & 4 & 2 \\ -9 & -2 & 6 & -12 & -7 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 0 & 6 \end{array}\right) \quad R_2 \to R_2 + 3R_1$$

$$\left(\begin{array}{cccc|c} 3 & 1 & -2 & 4 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 0 & 6 \end{array}\right) \quad R_4 \to R_4 - 2R_3$$

$$\longrightarrow \left(\begin{array}{cccc|c} 3 & 1 & -2 & 4 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right)$$

<u>row 4</u> : $2x_4 = 2 \Rightarrow \underline{x_4 = 1}$

<u>row 3</u> $x_3 - x_4 = 2 \Rightarrow x_3 = 3$

<u>row 2</u> : $\underline{x_2 = -1}$

<u>row 1</u> : $3x_1 + x_2 - 2x_3 + 4x_4 = 2$

$\Rightarrow 3x_1 - 1 - 2 \times 3 + 4 \times 1 = 2$

$\underline{x_1 = 5/3}$.

28.1.2 Definition (row echelon form)

A matrix is in *row echelon form* (r.e.f.) if each row after the first starts with *more* zeros than the previous row (or else rows at bottom of matrix are all zeros).

The Gauss algorithm converts any matrix to one in row echelon form. The 2 matrices are *equivalent*, that is, they have the same solution set.

$$A\underline{x} = \underline{b} \xrightarrow{\text{row op's}} \boxed{A'\underline{x} = \underline{b}'}$$

28.1.3 Elementary row operations

1. $r_i \leftrightarrow r_j$: swap rows i and j.
2. $r_i \rightarrow r_i - cr_j$: replace row i with (row i minus c times row j).
3. $r_i \rightarrow cr_i$:
replace row i with c times row i, where $c \neq 0$.

The Gauss algorithm uses only **1** and **2**.

28.2 Possible solutions for $Ax = \boldsymbol{b}$

Consider the r.e.f. of $[A|\boldsymbol{b}]$. Then we have three possibilities:

(1) *Exactly one* solution; here the r.e.f. gives each variable a single value, so the number of variables, n, equals the number of non-zero rows in the r.e.f.

(2) *No* solution; when one row of r.e.f. is $(0\,0\,\ldots\,d)$ with $d \neq 0$. We can't solve $0x_1 + 0x_2 + \cdots + 0x_m = d$ if $d \neq 0$; it says $0 = d$. In this case the system is said to be *inconsistent*.

(3) *Infinitely many* solutions; here the number of non-zero rows of the r.e.f. is *less* than the number of variables.

Note that a *homogeneous* system has $\boldsymbol{b} = \boldsymbol{0}$, i.e., all zero RHS. Then we always have at least the trivial solution, $x_i = 0$, $1 \leq i \leq n$.

28.2.1 Examples

$$
\begin{aligned}
x_1 + x_2 - x_3 &= 0 \\
2x_1 - x_2 &= 0 \\
4x_1 + x_2 - 2x_3 &= 1
\end{aligned}
$$

$$
\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 0 & 0 \\ 4 & 1 & -2 & 1 \end{array}\right)
$$

$$\Big\downarrow \text{ row op's.}$$

$$
\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \text{\# problem.}
$$

row 3 $\rightarrow$ $0x_1 + 0x_2 + 0x_3 = 1$

$\Rightarrow$ no solution exists.

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 2 \\ 2x_1 - 3x_2 + 7x_3 &= 6 \\ x_2 - x_3 &= 2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 2 & -3 & 7 & 6 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$$\Big\downarrow \text{row op's}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad * \text{ row of } 0\text{s}$$

$\Rightarrow$ infinitely many solutions.

row 2 $\Rightarrow$ $x_2 - x_3 = 2$

set $x_3 = t$, $x_2 = 2+t$.

row 1 $\Rightarrow$ $1x_1 - 2x_2 + 4x_3 = 2$

$\Rightarrow x_1 = 2(2+t) - 4t + 2$

$= 6 - 2t$.

28.3 Elementary matrices

An $n \times n$ matrix is called elementary if it can be obtained from the $n \times n$ identity matrix by performing one of the three elementary row operations.

For example, for 3 × 3 matrices, identity $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- A type 1 row operation is $R_2 \leftrightarrow R_3$ which corresponds to the elementary matrix
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
- A type 2 row operation is $R_2 \to R_2 - 3R_1$ which corresponds to the elementary matrix
$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- A type 3 row operation is $R_3 \to 5R_3$ which corresponds to the elementary matrix
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

In fact, applying an elementary row operation to any $n \times m$ matrix A is equivalent to multiplying A from the left by the corresponding elementary matrix.

For example,

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & 4 & 4 & 0 \\ 2 & -1 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & -1 & -3 & -3 \\ 1 & 4 & 4 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}.$$

$AA^{-1} = I = A^{-1}A$.

Elementary matrices are useful theoretical tools. Many proofs of fundamental results in linear algebra rely on these matrices and their properties.

For example, we can view the steps in determining the inverse of a square matrix as a sequence of operations involving elementary matrices. We look for a solution X to the matrix equation $\boxed{AX = I}$ by forming the augmented matrix $\boxed{(A|I)}$ and performing elementary row operations. For example, after performing three operations, we have really changed the equation to

$(A|I) \xrightarrow[\text{op's}]{\text{row}} (I|A^{-1})$

$$E_3E_2E_1AX = E_3E_2E_1I,$$

where E_1, E_2, E_3 are elementary matrices. On completion of the steps (say there are n of them), we reach

$$E_n \dots E_2E_1AX = IX = X = E_n \dots E_2E_1I,$$

which tells us that the inverse of A (usually denoted A^{-1}), if it exists, is nothing more than the product of elementary matrices $E_n \dots E_2E_1$.

In fact, if $A^{-1} = E_n \dots E_2E_1$, then the matrix A itself must be a product of inverses of elementary matrices $A = E_1^{-1}E_2^{-1} \dots E_n^{-1}$. It turns out, as we shall see, that the inverse of an elementary matrix is an elementary matrix. Hence if A is invertible, then it can be written as a product of elementary matrices.

28.3.1 Two important results regarding determinants

A^T = "transpose of A" rows ⟺ columns.

Two significant results regarding determinants are

$$\underline{\det(AB) = \det(A)\det(B)}, \quad \overline{\det(A) = \det(A^T)}$$

You should already be familiar with these results. They can be proved by the use of elementary matrices, by first establishing the results where A is an elementary matrix, and then generalising.

The proofs are beyond the scope of this course, but it is worth mentioning that the proofs make use of elementary matrices, hence demonstrating their importance.

28.3.2 Inverses of elementary matrices

It is a simple matter to verify that elementary matrices of type 1 (corresponding to the row operation of swapping two rows) square to the identity. In other words, the inverses of these matrices are just the matrices themselves. See, for example, that

type 1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that in section 30.1 we refer to these elementary matrices as permutation matrices.

Let us consider elementary matrices of type 2. It is straightforward to give the inverse of these matrices. Note the pattern for the following 3×3 matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix},$$

type 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix}.$$

Finally, the inverses of the type 3 elementary matrices are simply

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

type 3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/c \end{pmatrix}.$$

The significance here is that in general, the inverse of an elementary matrix is an elementary matrix of the same type.

type 2 only.

Say we need to perform three elementary row operations to obtain a r.e.f. of A. We can then write

$$E_3E_2E_1A = U,$$

where U is the r.e.f. of A. Since we know the inverses of all elementary matrices (indeed, they do exist), we can write

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U.$$

By observation, the matrix $L = E_1^{-1}E_2^{-1}E_3^{-1}$ is lower triangular with 1's on the main diagonal.

For the matrix $A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -2 \\ 2 & -2 & 10 \end{pmatrix}$, the two operations which give the r.e.f. are $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$, so the r.e.f of A can be expressed in terms of elementary matrices

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

Since we can easily invert these elementary matrices, we have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} U$$

$$\Rightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} U.$$

$A = L\ U$