29 *LU* decompositions

By the end of this section, you should be able to answer the following questions:

- How do you find an LU decomposition of a matrix?
- How do you use an LU decomposition to solve a system of equations?

29.1 Finding L and U

Given an $m \times n$ matrix A, we use the Gauss algorithm to find the r.e.f. U (which is also $m \times n$) for A.

Say *no* row interchanges are used, so there are only operations of the form $r_i \to r_i - cr_j$. Let c_{i1} be the multiple of the 1st row subtracted from the ith row, c_{i2} be the multiple of the 2nd row subtracted from the (new) ith row, etc., when finding U.

$R_i \to R_i - c_{ij} R_j$

Form the $m \times m$ lower triangular matrix:

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ c_{21} & 1 & 0 & \dots & 0 \\ c_{31} & c_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \dots & 1 \end{pmatrix}$$

Our main result is that indeed $A = LU$. This is what we call an LU decomposition of A.

* It is worth stressing that *it is only possible to find an LU decomposition if no row interchanges are used.* *

We remark that not every matrix has an LU decomposition. If, however, a matrix does have an LU decomposition, then

(square matrices)

$$\det A = \det L \det U = \det U.$$

- det of an upper/lower triangular matrix = product of entries on diagonal.
- order of row op's is important.

29.1.1 Example

Find the LU decomposition for A where $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \\ -2 & -6 & 8 \end{pmatrix}$, then calculate $\det A$.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - (-1)R_1$

$$\begin{pmatrix} 2 & 3 & -1 \\ 2 & -1 & 2 \\ -1 & -3 & 7 \end{pmatrix}$$

Use n in the (i,j) entry to record the row operation used to make that entry zero. i.e. n in the (i,j) entry $\iff$ row op. $R_i \to R_i - nR_j$ was used. (but entry is really zero)

$$R_3 \to R_3 - 3R_2 \to \begin{pmatrix} 2 & 3 & -1 \\ 2 & -1 & 2 \\ -1 & 3 & 1 \end{pmatrix}$$

$$\Rightarrow U = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$$

Check that $A = LU$

$$\det(A) = \underbrace{\det L}_{=1} \det U = 2 \times (-1) \times 1 = -2.$$

Summary of p190

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\overbrace{\begin{pmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \\ -2 & -6 & 8 \end{pmatrix}}^{A}$$

$$= \begin{pmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} - \text{r.e.f} = U$$

$$\Rightarrow A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}}_{\parallel \; L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}}\begin{pmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

29.2 Using an LU decomposition to solve systems of equations

$A = LU \Rightarrow L(U\underline{x}) = \underline{b}$

We can use this decomposition to solve $\boxed{A\boldsymbol{x} = \boldsymbol{b}}$ by first setting $\boldsymbol{y} = U\boldsymbol{x}$ and then solving $L\boldsymbol{y} = \boldsymbol{b}$ to obtain $\boldsymbol{y}$, and then solving $U\boldsymbol{x} = \boldsymbol{y}$ to obtain the solution $\boldsymbol{x}$.

Since L is lower triangular and U is in r.e.f., solving $L\boldsymbol{y} = \boldsymbol{b}$ (by forward substitution) and $U\boldsymbol{x} = \boldsymbol{y}$ (by back substitution) are both straightforward.

The advantage of this method is that we only need to compute L and U once. Then we can use them for many different $\boldsymbol{b}$, even when perhaps $\boldsymbol{b}_j$ depends upon earlier $\boldsymbol{b}$. This method also works if A is singular.

1 Set $\boxed{\underline{y} = U\underline{x}}$, solve $L\underline{y} = \underline{b}$ for $\underline{y}$

2 solve $U\underline{x} = \underline{y}$ for $\underline{x}$.

29.2.1 Example

Given $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \\ -2 & -6 & 8 \end{pmatrix}$, solve $A\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Same A as p190

$A\underline{x} = \underline{b} \Rightarrow L(U\underline{x}) = \underline{b}$

1 Set $U\underline{x} = \underline{y}$ & solve $L\underline{y} = \underline{b}$.

i.e. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

row 1 $\Rightarrow$ $\underline{y_1 = 1}$

row 2 $\Rightarrow$ $2y_1 + y_2 = 2 \Rightarrow \underline{y_2 = 0}$

row 3 $\Rightarrow$ $-y_1 + 3y_2 + y_3 = 3 \Rightarrow \underline{y_3 = 4}$

$\underline{y} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$

(2) Now solve $U\underline{x} = \underline{y}$.

$$\Rightarrow \begin{pmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

row 3 $\Rightarrow$ $\underline{x_3 = 4}$

row 2 $\Rightarrow$ $-x_2 + 2x_3 = 0 \rightarrow \underline{x_2 = 8}$

row 1 $\Rightarrow$ $2x_1 + 3x_2 - x_3 = 1$

$\Rightarrow 2x_1 + 24 - 4 = 1 \rightarrow \underline{x_1 = -19/2}$

sol. is $\underline{x} = \begin{pmatrix} -19/2 \\ 8 \\ 4 \end{pmatrix}$

Benefit of using LU:

computationally efficient for

- large systems
- $\boxed{\text{multiple systems (different RHS)}}$ *
- m×n systems.