Stewart CHAP 18.

3 Linear second order nonhomogeneous ODEs, method of undetermined coefficients

+ Revision.

By the end of this section, you should be able to answer the following questions:

- How do you apply the method of undetermined coefficients to solve a nonhomogeneous linear second order ODE?
- Under what conditions will the method work?

ODEs can be split into two classes: linear and non-linear. Non-linear ODEs are generally very difficult to solve. Linear ODEs are simpler because their solutions have general properties which facilitate working with them. There are also well established methods for solving many linear ODEs of practical significance.

A second order ODE is called linear if it can be written in the form

$$\boxed{y'' + p(x)y' + q(x)y = r(x).} \tag{3}$$

Any second order ODE which cannot be written in this form is called non-linear. Note that y and its derivatives appear linearly and p, q and r can be any functions.

Over the next few sections we study linear second order ODEs. The motivation for studying second order ODEs is twofold. Firstly they have applications in mechanics and electric circuit theory, so anyone studying either of these fields will most likely come across second order ODEs. Secondly, the theory of linear second order ODEs is very similar to that of higher order linear ODEs, so that the transition to studying higher order linear ODEs would not require too many new ideas.

Second order linear ODEs were introduced in MATH1052, and here we first recall some important results.

3.1 The superposition principle

If $r(x) = 0$ in equation (3), then we call the equation homogeneous. If $r(x) \neq 0$, the ODE is nonhomogeneous.

For any homogeneous linear equation, if y_1 and y_2 are solutions, so too is the linear combination $\boxed{Ay_1 + By_2}$. This is called the superposition principle. It is important to note that the superposition principle is not true for nonlinear equations and nonhomogeneous.

3.2 General solutions and initial value problems (homogeneous)

The general solution of a homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0$$

is of the form

$$y = Ay_1 + By_2,$$

that is, a linear combination of two linearly independent solutions with two arbitrary constants A and B.

An initial value problem consists of a homogeneous (in this case) linear second order ODE and two initial conditions

$$y(x_0) = K_0, \quad y'(x_0) = K_1.$$

3.3 Homogeneous ODEs with constant coefficients

Let a, b be constants. We look at solving the ODE

$$y'' + ay' + by = 0. \tag{4}$$

By assuming the solution is of the form $y = e^{\lambda x}$, we conclude that λ satisfies the quadratic

$$\lambda^2 + a\lambda + b = 0.$$

This quadratic is called the characteristic equation (or auxillary equation) of (4), the roots of which are given by

$$\lambda_\pm = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

The form of the general solution depends on the roots of the characteristic equation, summarised in the table below.

Roots	General Solution
real distinct λ_+, λ_-	$y = Ae^{\lambda_+ x} + Be^{\lambda_- x}$
single real $\lambda = \alpha$	$y = (A + Bx)e^{\alpha x}$
complex $\lambda_\pm = \beta \pm i\omega$	$y = e^{\beta x} A \cos \omega x + e^{\beta x} B \sin \omega x$

3.4 Method of undetermined coefficients

Now we consider equations of the form

$$y'' + p(x)y' + q(x)y = r(x), \quad r(x) \neq 0. \tag{5}$$

You should know from MATH1052 that the general solution on an open interval I is

$$\boxed{y = y_H + y_P,}$$

where y_H is the general solution of the homogeneous equation (with $r(x) = 0$) on I and y_P is a particular solution of (5) on I containing no arbitrary constants.

In what follows, we determine a solution to the homogenous equation, and then *try* a form (with *undetermined coefficients*) for the particular solution which looks like it will result in the function on the right hand side.

The method of undetermined coefficients, as presented here, only works for the constant coefficient case:

$$y'' + ay' + by = r(x),$$

and $r(x)$ contains exponentials, polynomials, sines and cosines, or sums and certain products of these functions.

Choose for y_P a form similar to $r(x)$, involving unknown coefficients. The coefficients are then determined by substituting y_P into the ODE.

$r_i(x)$	$g_i(x)$	$r_i(x)$	$g_i(x)$
$ke^{\gamma x}$	$ae^{\gamma x}$	$k\cos\omega x$, $k\sin\omega x$	$a\cos\omega x + b\sin\omega x$
$\sum_{i=0}^{N} k_i x^i$, $N = 0, 1, 2, \ldots$	$\sum_{i=0}^{N} a_i x^i$	$ke^{\alpha x}\cos\omega x$, $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(a\cos\omega x + b\sin\omega x)$

See Stewart pp 1154–1158

Summary

3.5 Rules for method of undetermined coefficients

We follow these basic steps.

1. Find a solution y_H to the corresponding homogeneous equation.
2. For $r(x) = r_1(x) + r_2(x) + \ldots + r_n(x)$, we first make a guess $g(x) = g_1(x) + g_2(x) + \ldots + g_n(x)$ for y_P, where the $g_i(x)$ correspond to the $r_i(x)$ entries in the table above.

modify guess

3. If a term $g_i(x)$ appears in y_H, replace $g_i(x)$ in the initial guess by $xg_i(x)$.
4. If any of the $xg_i(x)$ from step 3 appear in y_H, replace $xg_i(x)$ by $x^2g_i(x)$.
5. Substitute the modified guess $g(x)$ into the left hand side of the ODE and equate coefficients on both sides. Once you have worked out the coefficients, the guess $g(x)$ becomes y_P.

general solution $y = y_H + y_P$

3.6 Example: $y'' + 4y' + 4y = 8x^2$

For y_H: char. equ. $\lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow (\lambda+2)^2 \Rightarrow \lambda = -2$.

$\Rightarrow y_H = Ae^{-2x} + Bxe^{-2x}$

For y_P: guess $y_P = ax^2 + bx + c$:

$y_P' = 2ax + b$

$y_P'' = 2a$

$\Rightarrow y_P'' + 4y_P' + 4y_P = 2a + 4(2ax+b) + 4(ax^2+bx+c)$

$= 2a + 4b + 4c + x(8a + 4b) + x^2 4a$

$= RHS = 8x^2$

equate coefficients: x^2: $4a = 8 \Rightarrow a = 2$

x: $8 \times 2 + 4b = 0 \Rightarrow b = -4$

1: $2 \times 2 + 4 \times (-4) + 4c = 0 \Rightarrow c = 3$

$\Rightarrow$ gen. sol. is $y = Ae^{-2x} + Bxe^{-2x} + 2x^2 - 4x + 3$.

The method looks relatively simple, but there are a number of well known special cases which the rules deal with. Consider the following two examples:

3.7 Example: $y'' + y' - 2y = -3e^{-2x}$

general sol. $y = y_H + y_P$.

Check $y_H = Ae^{-2x} + Be^{x}$

For y_P: guess $y_P = ae^{-2x}$? ✗

won't work since e^{-2x} is part of y_H.

$\Longrightarrow$ modify guess $y_P = axe^{-2x}$.

Not in y_H $\longrightarrow$ ok.

$$\longrightarrow y = \underbrace{Ae^{-2x} + Be^{x}}_{y_H} + \underbrace{xe^{-2x}}_{y_P}.$$

3.8 Example: $y'' - 2y' + y = e^x$

Check: $y_H = Ae^x + B\underline{xe^x}$.

For y_p: guess $y_p = ae^x$? X

$\longrightarrow$ modify guess $y_p = a\underline{xe^x}$? X

$\longrightarrow$ $y_p = ax^2e^x$

sub. into LHS of ODE, equate coeffic's $\Longrightarrow a = 1/2$.

$\Longrightarrow$ gen. sol.

$$y = \underbrace{Ae^x + Bxe^x}_{y_H} + \underbrace{\tfrac{1}{2}x^2e^x}_{y_p}.$$

Note that if there are two terms on the right hand side we can handle each term separately. Consider the following extension of the previous example.

3.9 Extended example: $y'' - 2y' + y = e^x + \boxed{x}$

For x on RHS guess

$$y_{P2} = ax + b.$$

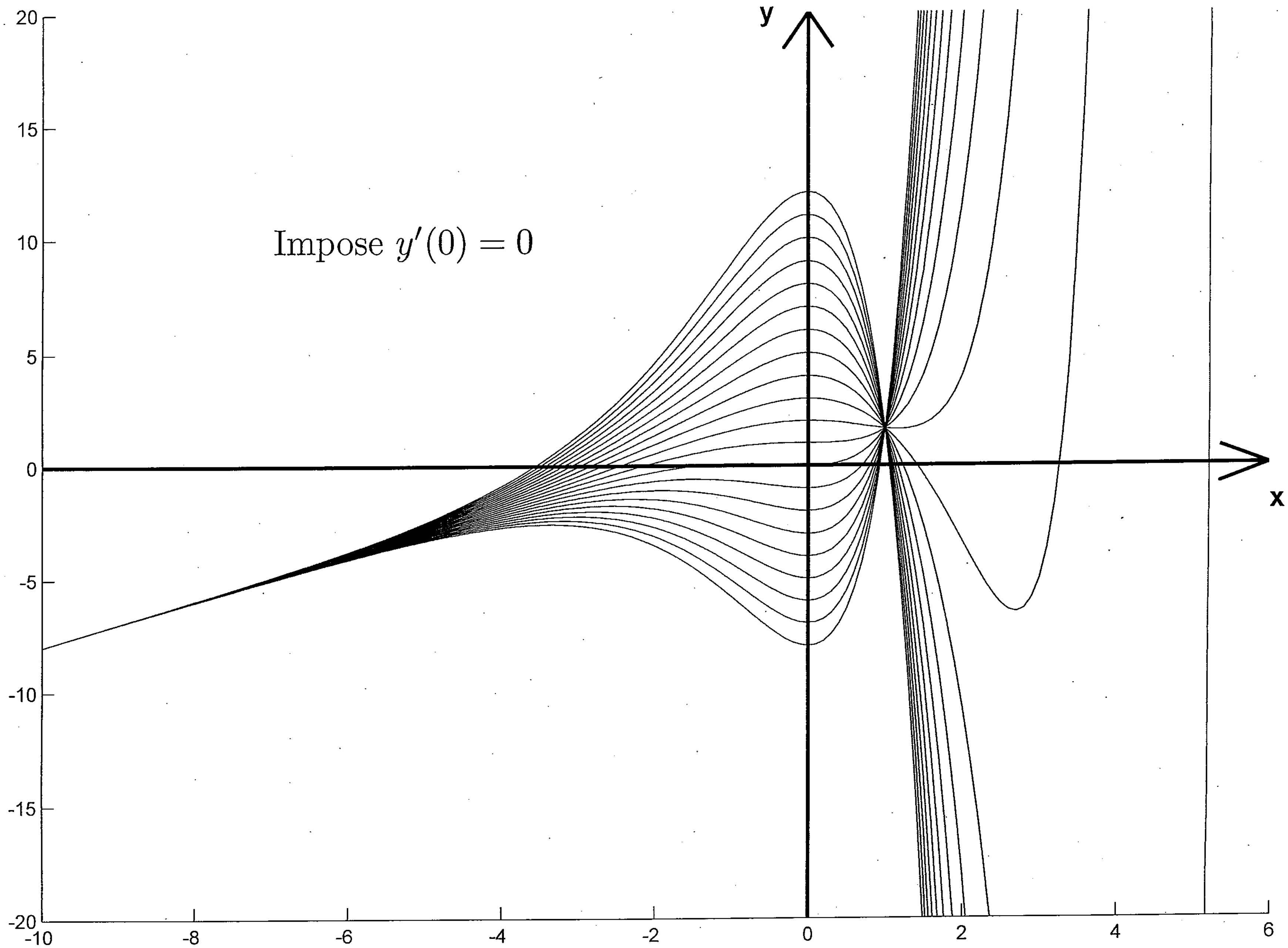
sub. into LHS

Check $y_{P2} = x + 2$.

gen. sol. is the

$$y = Ae^x + Bxe^x + \frac{1}{2}x^2e^x + x + 2.$$

If I.V.P., now impose the initial conditions.



Impose $y'(0) = 0$
y
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20
15
10
5
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-5
-10
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-20
-10
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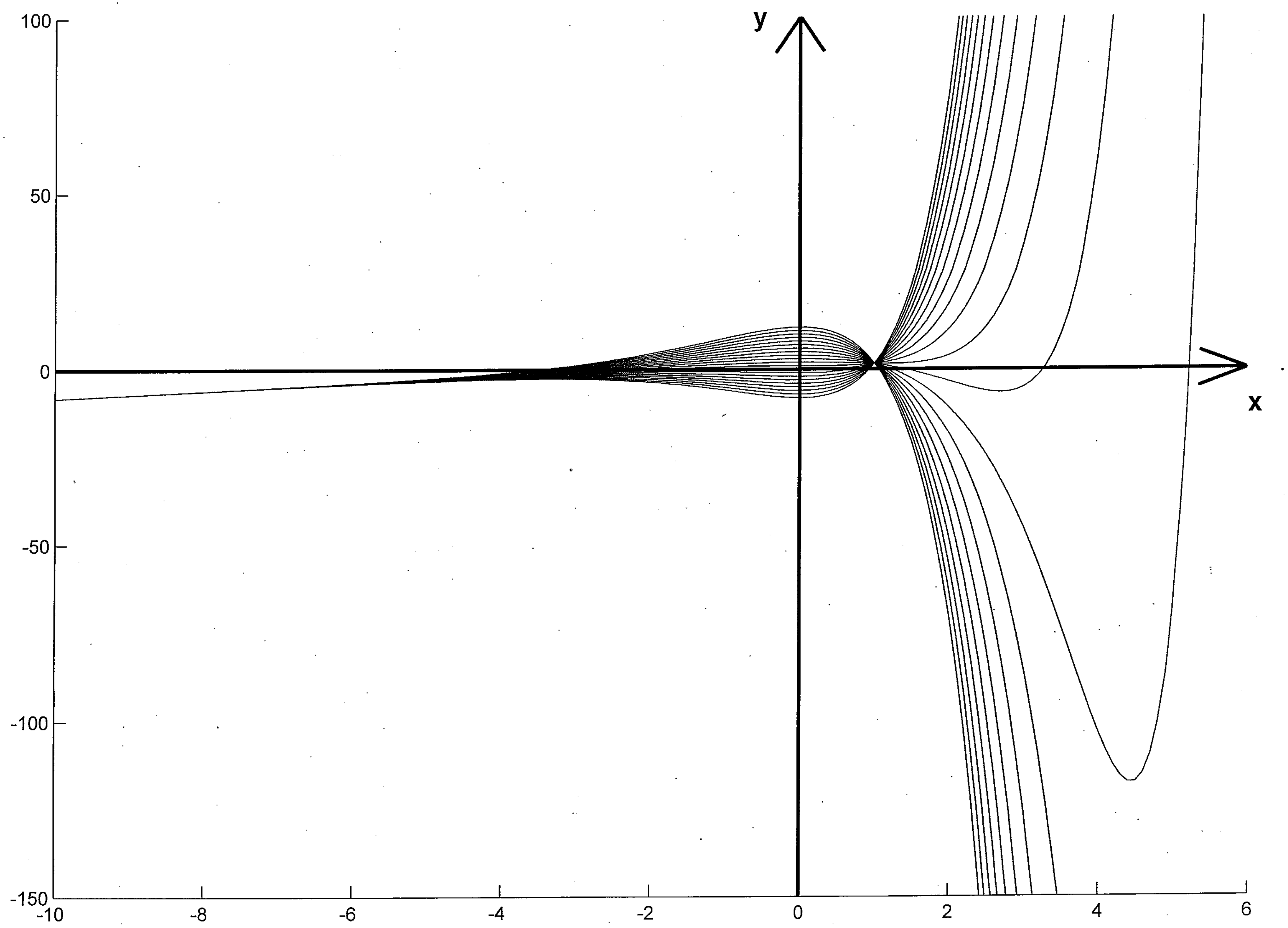
-6
-4
-2
0
2
4
6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-10
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-6
-4
-2
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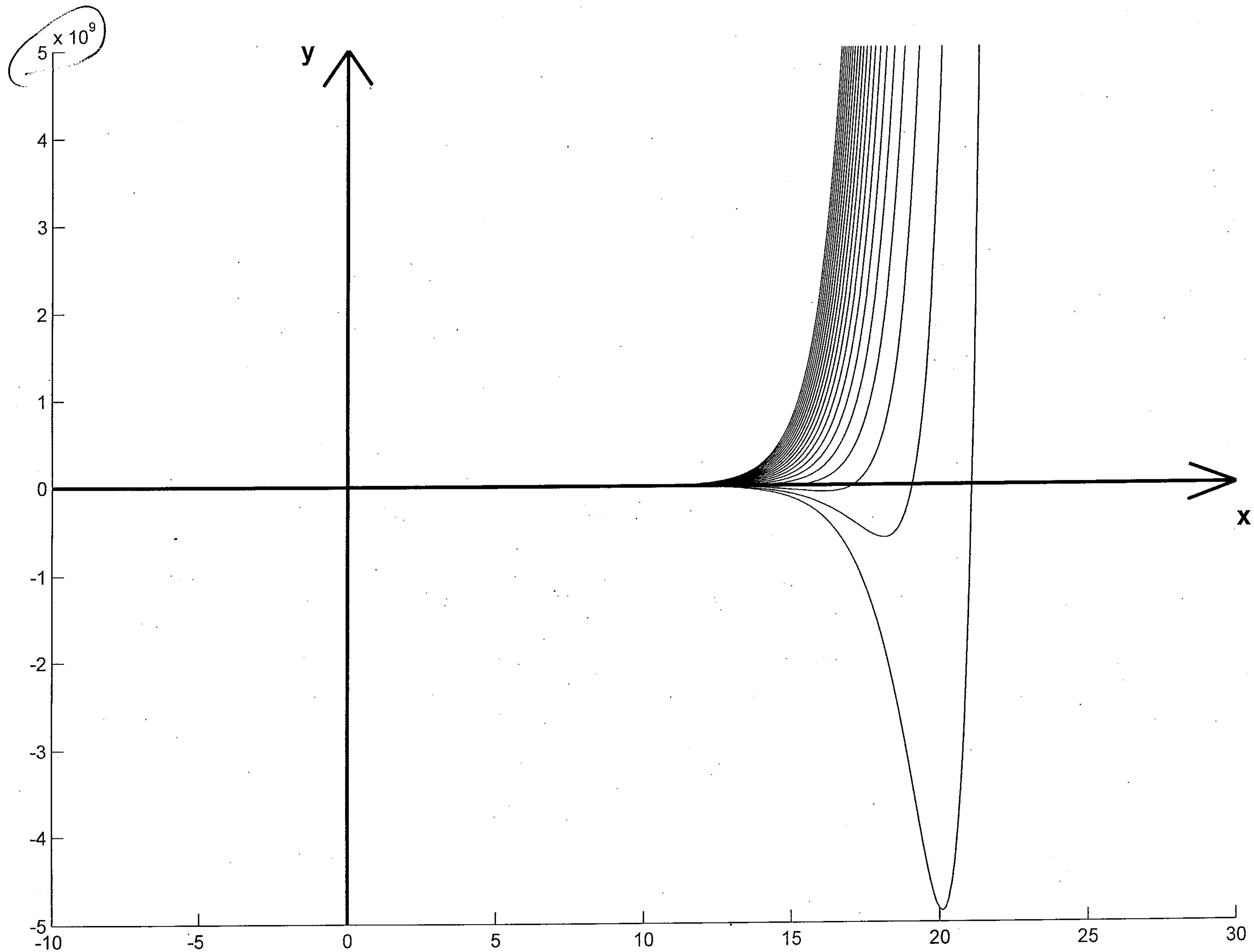
x 10^9
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15
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25
30