30 Permutation matrices and PLU decompositions

By the end of this section, you should be able to answer the following questions:

- How do you find a PLU decomposition of a matrix?
- How do you use a PLU decomposition to calculate a matrix determinant?

We mentioned in the last section that we can only find an LU decomposition if no row interchanges are needed to obtain the r.e.f. of a matrix. What if we *do* need row interchanges to get the r.e.f.?

30.1 Definition of permutation matrix

A *permutation matrix* is a matrix obtained from an identity matrix I by interchanging any 2 rows. ⟹ elementary matrices of "type 1".

Define $P_{k,\ell}^{(n)}$ as the permutation matrix obtained from the $n \times n$ identity I by swapping rows k and ℓ.

So, for example

$$P_{2,3}^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$P_{2,3}^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $P_{i,j}^{(k)} P_{i,j}^{(k)} = I^{(k)}$.

If A is $m \times n$, then $P_{k,\ell}^{(m)} A$ is a matrix obtained from A by swapping its rows k and ℓ.

If row interchanges are needed to get r.e.f. U from A, we could *first* rearrange all rows of A so that *no* interchanges are subsequently needed.
Say the system $A\boldsymbol{x} = \boldsymbol{b}$ is replaced by $A'\boldsymbol{x} = \boldsymbol{b}'$ after a series of row swaps, such that $A' = LU$. Then $A = PA'$ where P is a product of permutation matrices (maybe several).
Hence $A = PLU$.

30.2 Theorem (*PLU* decomposition)

Every $m \times n$ matrix A can be written in the form $A = PLU$ where P is a product of permutation matrices, L is an $m \times m$ lower triangular matrix with its main diagonal entries all 1, and U is an $m \times n$ r.e.f. matrix.

30.2.1 Example

Find a *PLU* decomposition of $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 4 \end{pmatrix}$.

$$R_2 \leftrightarrow R_3 \longrightarrow \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 0 & 0 \end{pmatrix} = A'$$

$$R_2 \to R_2 - 2R_1$$

$$\longrightarrow \begin{pmatrix} 1 & 0 \\ \textcircled{2} & 4 \\ 0 & 0 \end{pmatrix} = \text{r.e.f.}$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow A' = LU, \text{ but } A' = P_{23}^{(3)} A$$

$$\Rightarrow \underbrace{P_{23} P_{23}}_{I} A = P_{23} L U \Rightarrow \boxed{A = P_{23} L U}$$

<u>Note</u>: Initially it is not always obvious that we need to swap rows.

30.3 Determinants

This also gives an efficient way to find $\det(A)$, for a *square* matrix A.
If U is r.e.f. for A, found by using the Gauss algorithm on A, then

$$\det(A) = (-1)^N \det(U),$$

where N is the number of row interchanges used.

For: $\det(XY) = \det(X)\det(Y)$, so $\det(A) = \det(PLU) = \det(PL)\det(U)$.
But L is lower triangular with all 1s on main diagonal so $\det(L) = 1$.
PL is L with various rows interchanged.
Interchanging two rows of any determinant changes its sign. Hence $\det(PL) = \pm\det(L) = \pm 1$.

30.3.1 Example

Find a PLU decomposition of the matrix $A = \begin{pmatrix} 2 & 3 & -1 & 2 \\ -4 & -6 & 2 & 1 \\ 2 & 4 & 4 & -1 \\ 4 & 8 & 2 & 7 \end{pmatrix}$, then calculate its determinant. Note that it is not obvious in this case which rows to swap, so we treat it like a normal LU decomposition and then swap rows if required.

$R_2 \to R_2 - (-2)R_1$
$R_3 \to R_3 - 1R_1$
$R_4 \to R_4 - 2R_1$

$$\longrightarrow \begin{pmatrix} 2 & 3 & -1 & 2 \\ \textcircled{-2} & 0 & 0 & 5 \\ \textcircled{1} & 1 & 5 & -3 \\ \textcircled{2} & 2 & 4 & 3 \end{pmatrix} \quad R_2 \leftrightarrow R_4$$

$$\longrightarrow \begin{pmatrix} 2 & 3 & -1 & 2 \\ ② & 2 & 4 & 3 \\ ① & 1 & 5 & -3 \\ (-2) & 0 & 0 & 5 \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

$$\longrightarrow \begin{pmatrix} 2 & 3 & -1 & 2 \\ ① & 1 & 5 & -3 \\ ② & 2 & 4 & 3 \\ (-2) & 0 & 0 & 5 \end{pmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\longrightarrow \begin{pmatrix} 2 & 3 & -1 & 2 \\ ① & 1 & 5 & -3 \\ ② & ② & -6 & 9 \\ (-2) & 0 & 0 & 5 \end{pmatrix} = \text{r.e.f.}$$

$\Longrightarrow$ gives LU decomposition of

$$A' = P_{23} P_{24} A = LU$$

$$\Longrightarrow A = \boxed{P_{24} P_{23}} LU$$

$= P$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 3 & -1 & 2 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & -6 & 9 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det A = (-1)^2 \times \det U = 2 \times 1 \times (-6) \times 5 = -60$$

Summary of p195-196.

r.e.f. $U = E_4 \boxed{P_{23}P_{24}} E_3 E_2 E_1 A$

(E_1, E_2, E_3, E_4 type 2 elementary matrices)

e.g. In this case $E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}$

check $P_{23}P_{24}E_3 = \bar{E}_3 P_{23}P_{24}$

$\Rightarrow U = E_4 \bar{E}_3 \bar{E}_2 \bar{E}_1 P_{23}P_{24} A$

$A\underline{x} = \underline{b}$

$\Longrightarrow PLU\underline{x} = \underline{b}$

$\Longrightarrow \underbrace{P^{-1}P}_{=I} LU\underline{x} = P^{-1}\underline{b}$

$\Rightarrow L(U\underline{x}) = P^{-1}\underline{b} = \underline{b}'$

- set $\underline{y} = U\underline{x}$ solve $L\underline{y} = \underline{b}'$
- solve $U\underline{x} = \underline{y}'$