31 Eigenvalues and eigenvectors

By the end of this section, you should be able to answer the following questions:

- How do you find the eigenvalues and eigenvectors of a given square matrix?
- What are some simple properties of eigenvalues and eigenvectors?
- Prove that the eigenvectors corresponding to distinct eigenvalues are linearly independent.

A great deal of this section should be familiar to you. We start by recalling some results on vector spaces associated with matrices.

Review MATH1051

$V \subseteq \mathbb{R}^n$ s.t. $\underline{0} \in V$

$\underline{v}+\underline{w} \in V$ for $\underline{v}, \underline{w} \in V$

$k\underline{v} \in V$, $k \in \mathbb{R}$.

31.1 Column space, row space, rank, nullity

For any $m \times n$ real matrix A, the *null space* of A is the vector space

$$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

The dimension of $N(A)$ is called the *nullity* of A, denoted nullity(A).

The column space of an $m \times n$ matrix A is the space spanned by the column vectors of A ($\subseteq \mathbb{R}^m$). The dimension of the column space of A is called the *rank* of A, denoted rank(A). This coincides with the number of non-zero rows in the r.e.f. of A.

The row space of an $m \times n$ matrix A is the space spanned by the row vectors of A ($\subseteq \mathbb{R}^n$). A basis is given by the non-zero rows in the r.e.f. of A. The dimension of the row space is also given by the rank of A.

Note that the row space of A^T = column space of A.

For $m \times n$ matrices,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Def'n $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ is linearly independent

$\iff$ If $a_1\underline{v}_1 + a_2\underline{v}_2 + \cdots + a_n\underline{v}_n = \underline{0}$

then $a_1 = a_2 = \cdots = a_n = 0$

31.2 Non-singular matrices

For $n \times n$ square matrix A, we have several conditions for the existence of A^{-1}.

For $n \times n$ matrix A, the following are equivalent:

$AA^{-1} = A^{-1}A = I$.

1. A is non-singular. i.e. A^{-1} exists.
2. $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution $\boldsymbol{x} = \mathbf{0}$.
3. If U is a r.e.f. for A, then U has no row of all zeros.
4. $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every n-dimensional column vector $\boldsymbol{b}$.
5. $\det(A) \neq 0$.
6. The columns of A are linearly independent.
7. The rows of A are linearly independent.
8. $\text{nullity}(A) = 0$.
9. $\text{rank}(A) = \text{n}$.

31.3 Eigenvalues and eigenvectors

Let A be a square matrix. Then an *eigenvector* of A is a vector $\boldsymbol{v} \neq \mathbf{0}$ such that

$$A\boldsymbol{v} = \lambda \boldsymbol{v},$$

for some scalar λ.
The scalar λ is called the corresponding *eigenvalue*.

If $\boldsymbol{v}$ is an eigenvector of A, then so is $t\boldsymbol{v}$ for any scalar $t \neq 0$.

Recall if λ is an eigenvalue of A, with corresponding eigenvector $\boldsymbol{v}$, then $A\boldsymbol{v} = \lambda\boldsymbol{v} = \lambda I\boldsymbol{v}$, so $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$. Hence $\boldsymbol{x} = \boldsymbol{v}$ is a non-trivial solution to the homogeneous system of equations $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$, and conversely, if there is a non-trivial solution then λ is an eigenvalue of A. Thus:

λ is an eigenvalue of A $\Leftrightarrow A\underline{x} = \lambda \underline{x}$
if and only if $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has a non-trivial solution
if and only if $A - \lambda I$ is singular
if and only if $\det(A - \lambda I) = 0$. *

compare with 1, 2, 5 above

For an $n \times n$ matrix A, $\det(A - \lambda I)$ is a polynomial of degree n in λ, called the *characteristic polynomial* of A.

The equation $\det(A - \lambda I) = 0$ is the *characteristic equation* of A.

Eigenvalues λ may be complex numbers, and the eigenvectors $\boldsymbol{v}$ may have complex components, even for real matrices A.

To find the eigenvalues and eigenvectors, do the following:

1. Find the roots of the characteristic polynomial, $\det(A - \lambda I) = 0$. These are the eigenvalues.
2. For each eigenvalue λ, find all $\boldsymbol{v}$ satisfying $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$. These are the eigenvectors. The vector space spanned by the eigenvectors corresponding to each eigenvalue is called the *eigenspace* associated to λ.

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

31.3.1 Example

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$.

$$\det(A-\lambda I) = \det\begin{pmatrix} -3-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -3-\lambda \end{pmatrix}$$

$$= -(3+\lambda)\{(2+\lambda)(3+\lambda)-1\} - 1\{-(3+\lambda)\}.$$

$$= -(3+\lambda)\{\underline{6+5\lambda+\lambda^2-1}, -1\}.$$

$$= -(3+\lambda)\{\lambda^2+5\lambda+4\}.$$

$$= -(3+\lambda)(4+\lambda)(1+\lambda) = 0$$

$$\Rightarrow \lambda = -3, -4, -1.$$

$(A-\lambda I)\underline{v} = \underline{0}$

$\underline{\lambda = -3}$: $(A+3I)\underline{v} = \underline{0}$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow b = 0 \ \& \ a+c = 0 \Rightarrow c = -a$$

$$\Rightarrow \begin{pmatrix} a \\ 0 \\ -a \end{pmatrix} \text{ or } a\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$(A+4I)\underline{v}=\underline{0}$

$\underline{\lambda=-4}$: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \left.\begin{matrix} a+b=0 \\ b+c=0 \end{matrix}\right\} \Rightarrow a=-b=c$

$\Rightarrow \begin{pmatrix} c \\ -c \\ c \end{pmatrix}$ or just $c\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Check $\lambda=-1 \to c\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

__Note:__ Can never have $\underline{0}$ as an eigenvector.

For $n = 2, 3$, we can solve the characteristic equation to get eigenvalues. For $n \geq 4$ there are better numerical methods.

31.4 Simple properties

For a square matrix A:

1. A and A^T have the same eigenvalues.

$\det(A^T - \lambda I) = \det((A-\lambda I)^T)$

$= \det(A - \lambda I)$

diagonal

2. A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

logic from p198

A singular $\iff$ $A\underline{x} = \underline{0}$ has nontrivial sol.

$\iff$ $\lambda = 0$ is an eigenvalue.

3. If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2, and $1/\lambda$ is an eigenvalue of A^{-1} when A is non-singular.

4. If λ is an eigenvalue of A, then $\lambda - m$ is an eigenvalue of $A - mI$, for any scalar m.

INDUCTION

Statement S_k $k = 1, 2, 3, \ldots$

Suppose the following hold:

1. S_1 is true
2. $\forall n \geq 1$, if S_n is true then S_{n+1} is true.

Then the statement
"$\forall k \geq 1$, S_k" is true.

In proof... ① Show S_1 is true
② Show that if S_n is true then S_{n+1} must be true $\forall n \geq 1$
$\Longrightarrow$ result.

31.5 Eigenvectors corresponding to distinct eigenvalues are linearly independent

If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of A, with corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ (such that $\boldsymbol{v}_i$ corresponds to λ_i), then the set of eigenvectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is linearly independent.

Use induction. • $k=1$, $\{\underline{v}_1\}$ is l.i.

• assume true for n such that $1 \le n < k$.

(i.e. assume $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ is l.i.)

Set $a_1 \underline{v}_1 + a_2 \underline{v}_2 + \ldots + a_n \underline{v}_n + a_{n+1} \underline{v}_{n+1} = \underline{0}$ (*)

(mult. (*) by A)

$\Rightarrow a_1 \lambda_1 \underline{v}_1 + a_2 \lambda_2 \underline{v}_2 + \ldots + a_n \lambda_n \underline{v}_n + a_{n+1} \lambda_{n+1} \underline{v}_{n+1} = \underline{0}$ (1)

(mult. (*) by $\lambda_{n+1} \neq 0$)

$\Rightarrow a_1 \lambda_{n+1} \underline{v}_1 + \ldots + a_{n+1} \lambda_{n+1} \underline{v}_{n+1} = \underline{0}$ (2).

(2) − (1) $\Rightarrow \underline{a_1(\lambda_{n+1} - \lambda_1)} \underline{v}_1 + \ldots + \underline{a_n(\lambda_{n+1} - \lambda_n)} \underline{v}_n = \underline{0}$

Since $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is l.i.

$\Rightarrow a_1(\lambda_{n+1} - \lambda_1) = \ldots = a_n(\lambda_{n+1} - \lambda_n) = 0$

$\Rightarrow a_1 = a_2 = \ldots = a_n = 0$.

Equ. (*) $\Rightarrow a_{n+1} \underline{v}_{n+1} = \underline{0} \Rightarrow a_{n+1} = 0$.

$\Rightarrow \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n, \underline{v}_{n+1}\}$ is l.i.

$\Rightarrow$ result by induction.