# 32 Diagonalisation

By the end of this section, you should be able to answer the following questions:

- How do you find a matrix $P$ which diagonalises a given matrix $A$?
- How do you determine if $A$ is diagonalisable?
- What are two applications of diagonalisation?

A square matrix $A$ is *diagonalisable* if there is a non-singular matrix $P$ such that $P^{-1}AP$ is a diagonal matrix. Here we consider the question: given a matrix, is it diagonalisable? If so, how do we find $P$? $A\underline{v} = \lambda \underline{v}$

The secret to constructing such a $P$ is to let the columns of $P$ be the eigenvectors of $A$. We immediately have that $\boxed{AP = PD}$ where $D$ is a diagonal matrix with eigenvalues on the diagonal. We know from section 31.2 on page 198 that $P$ is invertible if and only if the columns of $P$ are linearly independent. Hence, we have the following result: $1 \leftrightarrow 6$.

The $n \times n$ matrix $A$ is diagonalisable if and only if $A$ has $n$ linearly independent eigenvectors.

Is the matrix $A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$ diagonalisable?

eigenvalues $-3, -4, -1$. (pages 199–200)

3 distinct eigenvalues $\Rightarrow$ 3 l.i. eigenvectors

$\Rightarrow A$ is diagonalisable.

form matrix $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix}$

$\uparrow \lambda=-3 \quad \uparrow \lambda=-4 \quad \uparrow \lambda=-1$.

$P^{-1}$ exists (col's of $P$ are l.i.)

$\Rightarrow P^{-1}AP = D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

## 32.1 Similar matrices

Two matrices $A$ and $B$ are *similar* if there is a non-singular matrix $P$ such that $B = P^{-1}AP$.

The statements "$A$ is diagonalisable" and "$A$ is similar to a diagonal matrix" are equivalent.

### 32.1.1 Theorem (similar matrices)

Similar matrices have the same eigenvalues.

In fact, if $B = P^{-1}AP$ and $\boldsymbol{v}$ is an eigenvector of $A$ corresponding to eigenvalue $\lambda$, then $P^{-1}\boldsymbol{v}$ is an eigenvector of $B$ corresponding to eigenvalue $\lambda$. This is because

$$\begin{aligned} B(P^{-1}\boldsymbol{v}) &= (P^{-1}AP)P^{-1}\boldsymbol{v} \\ &= P^{-1}(A\boldsymbol{v}) \\ &= P^{-1}(\lambda\boldsymbol{v}) \\ &= \lambda(P^{-1}\boldsymbol{v}) \end{aligned}$$

## 32.2 A closer look at the diagonal matrix

Let the matrix $A$ be $n \times n$ with $n$ linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. Let

$$P = (\boldsymbol{v}_1 | \ldots | \boldsymbol{v}_n)$$

be the $n \times n$ matrix whose columns are the eigenvectors. Then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix},$$

the diagonal matrix with the eigenvalues down the main diagonal. The important point here is the order in which the eigenvalues appear. They correspond to the order in which the associated eigenvectors appear in the columns of $P$.

3×3 case

$$A\left(\underset{\sim}{v}_1 \middle| \underset{\sim}{v}_2 \middle| \underset{\sim}{v}_3\right) = \left(A\underset{\sim}{v}_1 \middle| A\underset{\sim}{v}_2 \middle| A\underset{\sim}{v}_3\right)$$

$$= \left(\lambda_1\underset{\sim}{v}_1 \middle| \lambda_2\underset{\sim}{v}_2 \middle| \lambda_3\underset{\sim}{v}_3\right)$$

$$= \left(\underset{\sim}{v}_1 \middle| \underset{\sim}{v}_2 \middle| \underset{\sim}{v}_3\right)\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$\Rightarrow AP = PD$

## 32.3 Diagonalisability

We know that an $n \times n$ matrix $A$ is diagonalisable if and only if $A$ has $n$ linearly independent eigenvectors.

Now say $\lambda_1, \ldots, \lambda_m$ are *distinct* eigenvalues of $A$, with corresponding eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$. Then we have also seen that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are linearly independent.

Hence if $A$ is $n \times n$ with $n$ distinct eigenvalues, then $A$ is diagonalisable.

The question remains, if $A$ has fewer than $n$ distinct eigenvalues, how do we know if $A$ is diagonalisable?

### 32.3.1 Example

Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Easy to see the characteristic equation of both $A$ and $B$ is $(2-\lambda)(1-\lambda)^2 = 0$, so $\lambda = 2, 1, 1$.

$A$ $\quad \underline{\lambda = 2}$ :

$\underline{(A - \lambda I)\underline{v} = \underline{0}}$ $\quad \begin{pmatrix} 0 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Longrightarrow b = c = 0$, no condition on $a$

$\Longrightarrow \underline{v}_1 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{\lambda = 1}$ : $\quad \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Longrightarrow a = -b - 3c$

$\Longrightarrow \begin{pmatrix} -b-3c \\ b \\ c \end{pmatrix} = b\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{v_2} + c\underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}_{v_3}$

$\Longrightarrow A$ is diagonalisable.

B $\underline{\lambda = 2}$ $\Longrightarrow$ $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{\lambda = 1}$ : $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow c = 0$ , $a + b = 0 \Rightarrow \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

only 2 l.i. eigenvectors

$\Longrightarrow$ B is NOT diagonalisable

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<u>algebraic mult.</u> (char. poly. $(2-\lambda)(1-\lambda)^2$ )

For A & B $\lambda = 2 \Longrightarrow$ a.m. $= 1$

$\lambda = 1 \Longrightarrow$ a.m. $= 2$

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<u>geometric mult.</u>

For A $\lambda = 2 \Rightarrow$ g.m. $= 1$, $\lambda = 1 \Rightarrow$ g.m. $= 2$

B $\lambda = 2 \Rightarrow$ g.m. $= 1$, $\lambda = 1 \Rightarrow$ g.m. $= 1$

Eigenspace $E_\lambda$ : vector space spanned by l.i. eigenvectors corresponding to eigenvalue $\lambda$.

## 32.4 Algebraic and geometric multiplicity

If we are only interested in finding out whether or not a matrix is diagonalisable, then we need to know the dimension of each eigenspace. There is one theorem (which we will not prove!) that states:

If $\lambda_i$ is an eigenvalue, then the dimension of the corresponding eigenspace cannot be greater than the number of times $(\lambda - \lambda_i)$ appears as a factor in the characteristic polynomial.

We often use the following terminology:

- The *geometric multiplicity* of the eigenvalue $\lambda_i$ is the dimension of the eigenspace corresponding to $\lambda_i$.
- The *algebraic multiplicity* of the eigenvalue $\lambda_i$ is the number of times $(\lambda - \lambda_i)$ appears as a factor in the characteristic polynomial.

The main result is the following:

> A square matrix is diagonalisable if and only if the geometric and algebraic multiplicities are equal for every eigenvalue.

Note that the geometric multiplicity of $\lambda_i$ is equal to nullity$(A - \lambda_i I)$. If $A$ is $n \times n$, then the result at the bottom of page 197 tells us that

$$\text{nullity}(A - \lambda_i I) = n - \text{rank}(A - \lambda_i I).$$

In practice, we can determine the geometric multiplicity of $\lambda_i$ by subtracting the number of non-zero rows in the r.e.f. of $(A - \lambda_i)$ from $n$. We then compare this number with the number of factors of $(\lambda - \lambda_i)$ to determine whether or not $A$ is diagonalisable.

One of many corollaries to this result is that the geometric multiplicities of $A$ and $A^T$ are equal.

e.g. $\frac{dx_1}{dt} = 2x_1 + 2x_2$

$\frac{dx_2}{dt} = 2x_1 - x_2$

$\dot{\underline{x}} = A\underline{x} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

## 32.5 Applications of diagonalisability

### 32.5.1 Systems of differential equations

For a system of coupled differential equations which can be written in matrix form as

$$\dot{x} = Ax$$

$\dot{\underline{x}} = PDP^{-1}\underline{x}$.

$\Rightarrow (P^{-1}\dot{\underline{x}}) = D(P^{-1}\underline{x})$

(where $x = (x_1, \dots, x_n)^T$, $\dot{x} = (\dot{x}_1, \dots, \dot{x}_n)^T$), if $A$ can be diagonalised, say $\boxed{P^{-1}AP = D}$ with $D$ diagonal, then make the substitution $\boxed{x = Py.}$ This yields

$$\boxed{\dot{y} = Dy}$$

set $\underline{y} = P^{-1}\underline{x}$

& $\dot{\underline{y}} = P^{-1}\dot{\underline{x}}$

which is easily solved.

$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\Rightarrow$ $\dot{y}_1 = \lambda_1 y_1$

$\dot{y}_2 = \lambda_2 y_2$

### 32.5.2 Matrix powers

If $A$ is diagonalisable, say $P^{-1}AP = D$ with $D$ diagonal, then

$$A^n = PD^nP^{-1}.$$

This gives an easy way to calculate $A^n$.

$A^n = (PDP^{-1})^n = \underbrace{PD\cancel{P^{-1}}\cancel{P}D\cancel{P^{-1}} \dots \cancel{P}DP^{-1}}_{n \text{ times}}$

$= PD^nP^{-1}$.

$D^n = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}$

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$A\underline{x} = \underline{b}$

$\Rightarrow PDP^{-1}\underline{x} = \underline{b}$.

$\Rightarrow D(P^{-1}\underline{x}) = P^{-1}\underline{b}$.

set $\underline{y} = P^{-1}\underline{x}$.

& solve $D\underline{y} = \underline{b}'$

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