4 Variation of parameters

By the end of this section, you should be able to answer the following questions:

- Under what conditions does the method work?
- What functions need to be determined first before using the method?
- How do you use the variation of parameters method to solve a nonhomogeneous linear second order ODE?

The method of undetermined coefficients is very easy to apply, but only works for constant coefficients with certain $r(x)$. In the case

general solution

$$y'' + p(x)y' + q(x)y = r(x) \qquad y = y_H + y_P.$$

has arbitrary coefficient functions p, q, r, the variation of parameters works all the time. The process is the following:

$$W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

- Solve $y'' + p(x)y' + q(x)y = 0$ to obtain a basis of solutions y_1, y_2 and set $\underline{W = y_1y_2' - y_1'y_2}$ (this quantity is known as the Wronskian of the solutions y_1 and y_2). There is a result that states that $W \neq 0$ if and only if y_1 and y_2 are linearly independent.

  $y_H = Ay_1 + By_2$

- Set $\underline{y_P = u(x)y_1(x) + v(x)y_2(x)}$ and substitute into the ODE. We also impose the condition $\underline{u'y_1 + v'y_2 = 0.}$ We have the freedom to impose this extra arbitrary condition because we have two functions (u and v) and only one equation they need to satisfy arising from the ODE.
- We obtain
$$u(x) = -\int \frac{y_2 r}{W}dx, \quad v(x) = \int \frac{y_1 r}{W}dx.$$

This approach is a variant of the method of Reduction of Order, which prescribes that we take a solution, say y_1 of the associated homogeneous equation and seek a particular solution of the form $\underline{y_p = U(x)y1.}$ or $y_p = V(x)y_2$

e.g. / 3rd order

$$W = \det\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}$$

4.1 Derive the formulae for $u(x)$ and $v(x)$ in the variation of parameters

general solution $y = y_H + y_P$

$$y_H = Ay_1 + By_2$$

Set $y_P = uy_1 + vy_2$

$$y_P' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

$$y_P'' = u''y_1 + 2u'y_1' + uy_1'' + v''y_2 + 2v'y_2' + vy_2''$$

$$y_P'' + py_P' + qy_P = u''y_1 + 2u'y_1' + uy_1'' + v''y_2 + 2v'y_2' + vy_2'' + pu'y_1 + puy_1' + pv'y_2 + pvy_2' + quy_1 + qvy_2$$

$$= u''y_1 + u'(2y_1' + py_1) + u\underbrace{(y_1'' + py_1' + qy_1)}_{=0} + v''y_2 + v'(2y_2' + py_2) + v\underbrace{(y_2'' + py_2' + qy_2)}_{=0}$$

impose the cunningly chosen condition

$$u'y_1 + v'y_2 = 0 \quad \text{--- (1)}$$

$$\Rightarrow u''y_1 + u'y_1' + v''y_2 + v'y_2' = 0$$

$$\Rightarrow y_P'' + py_P' + qy_P = u'y_1' + v'y_2' = r \quad \text{--- (2)}$$

(1) & (2) $\Rightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ r \end{pmatrix}$$

$$\Rightarrow u' = -\frac{y_2 r}{W}, \quad v' = \frac{y_1 r}{W}$$

& integrate.

4.2 Example: $y'' - 4y' + 5y = 2e^{2x}/\sin x$

$\underline{y_H}$: $\lambda^2 - 4\lambda + 5 = 0$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\Rightarrow \boxed{y_H = A\underbrace{e^{2x}\cos x}_{y_1} + B\underbrace{e^{2x}\sin x}_{y_2}}$$

$\underline{y_p}$: Set $y_p = u y_1 + v y_2$

where $u = -\int \frac{y_2 r}{w}\,dx$, $v = \int \frac{y_1 r}{w}\,dx$.

$$W = y_1 y_2' - y_1' y_2 = e^{2x}\cos x\left(\cancel{2e^{2x}\sin x} + e^{2x}\cos x\right) - \left(\cancel{2e^{2x}\cos x} - e^{2x}\sin x\right)e^{2x}\sin x$$

$$= e^{4x}(\cos^2 x + \sin^2 x) = e^{4x}.$$

$$\Rightarrow u = -\int \cancel{e^{2x}}\cancel{\sin x}\cdot \frac{2\cancel{e^{2x}}}{\cancel{\sin x}} \cdot \frac{1}{\cancel{e^{4x}}}\,dx = -2\int dx = -2x$$

$$\& \; v = \int \cancel{e^{2x}}\cos x \cdot \frac{2\cancel{e^{2x}}}{\sin x} \cdot \frac{1}{\cancel{e^{4x}}}\,dx = 2\int \frac{\cos x}{\sin x}\,dx.$$

$$= 2\ln|\sin x|$$

$$\Rightarrow \boxed{y_p = -2xe^{2x}\cos x + 2\ln|\sin x| \cdot e^{2x}\sin x.}$$

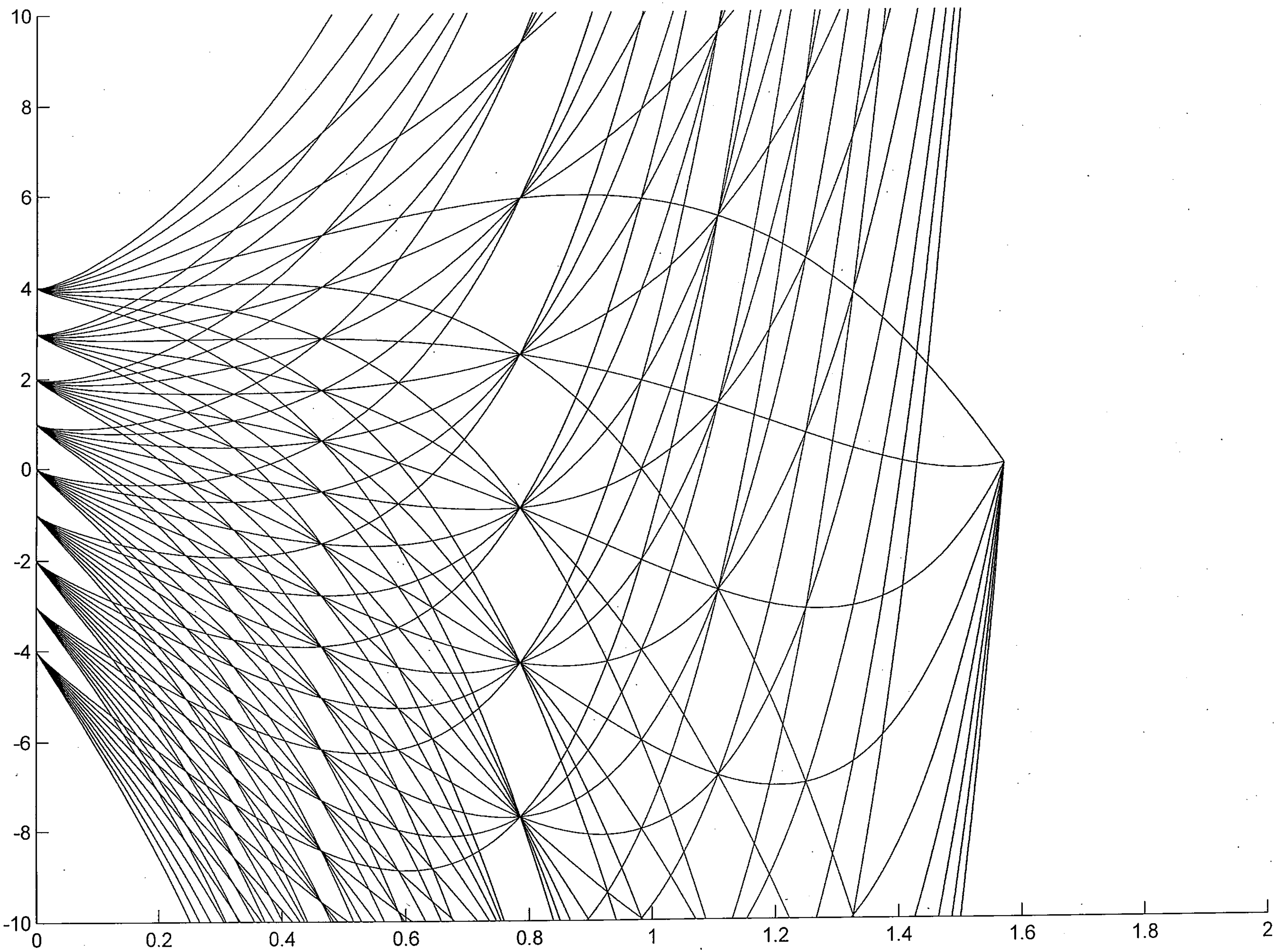
& general solution $y = y_H + y_p$ *

$y_p (= u y_1 + v y_2)$

$\Rightarrow \; = (u + C_1)y_1 + (v + C_2)y_2$

$= \underline{u y_1 + v y_2} + C_1 y_1 + C_2 y_2$

(C_1, C_2 are constants of integration).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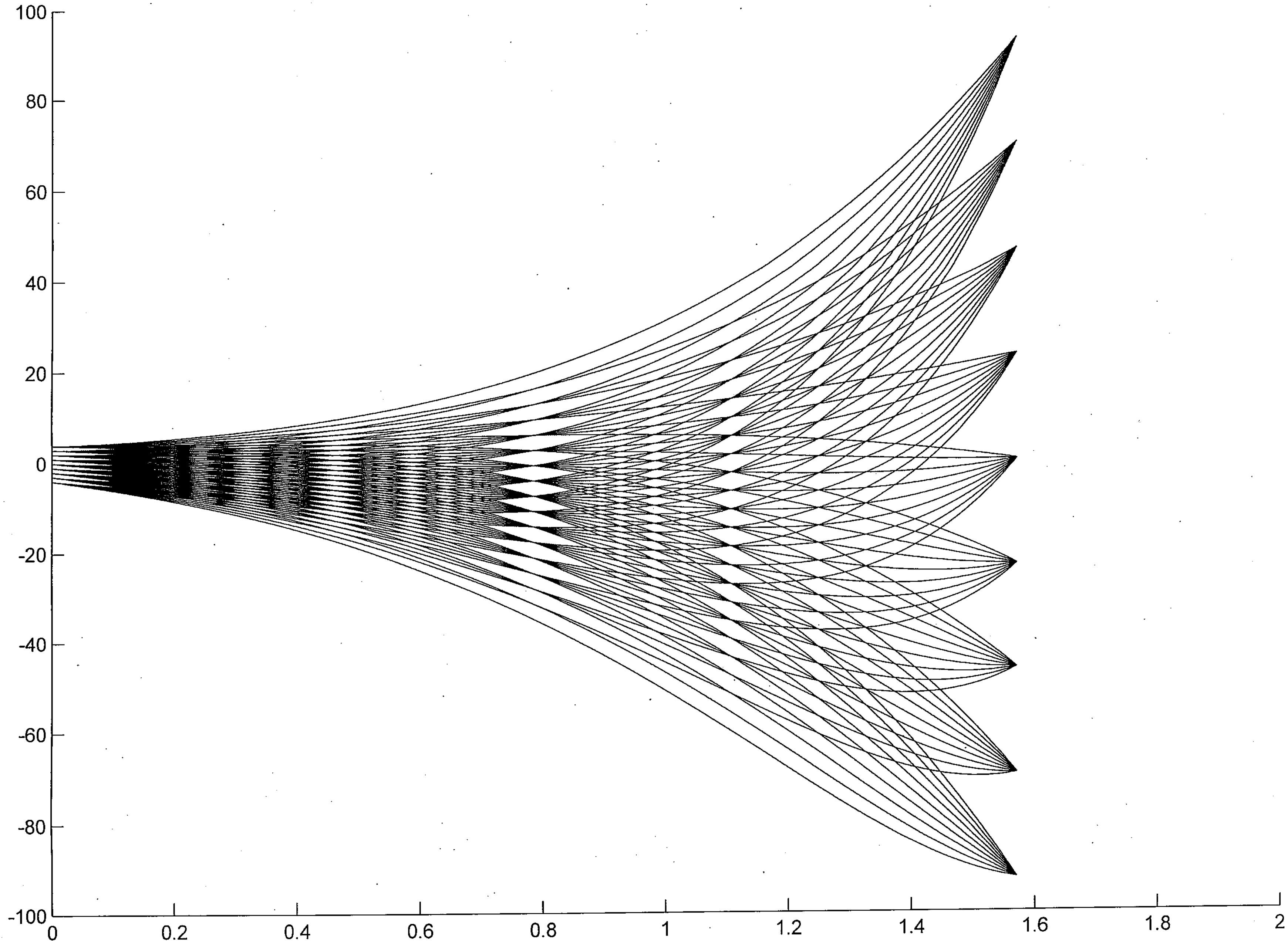

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cosec $2x$.

4.3 Example: $y'' + 4y = \csc 2x$ $= \frac{1}{\sin 2x} = r$

Check $\boxed{y_H = A \underbrace{\cos 2x}_{y_1} + B \underbrace{\sin 2x}_{y_2}}$

Check $W = y_1 y_2' - y_1' y_2 = 2$.

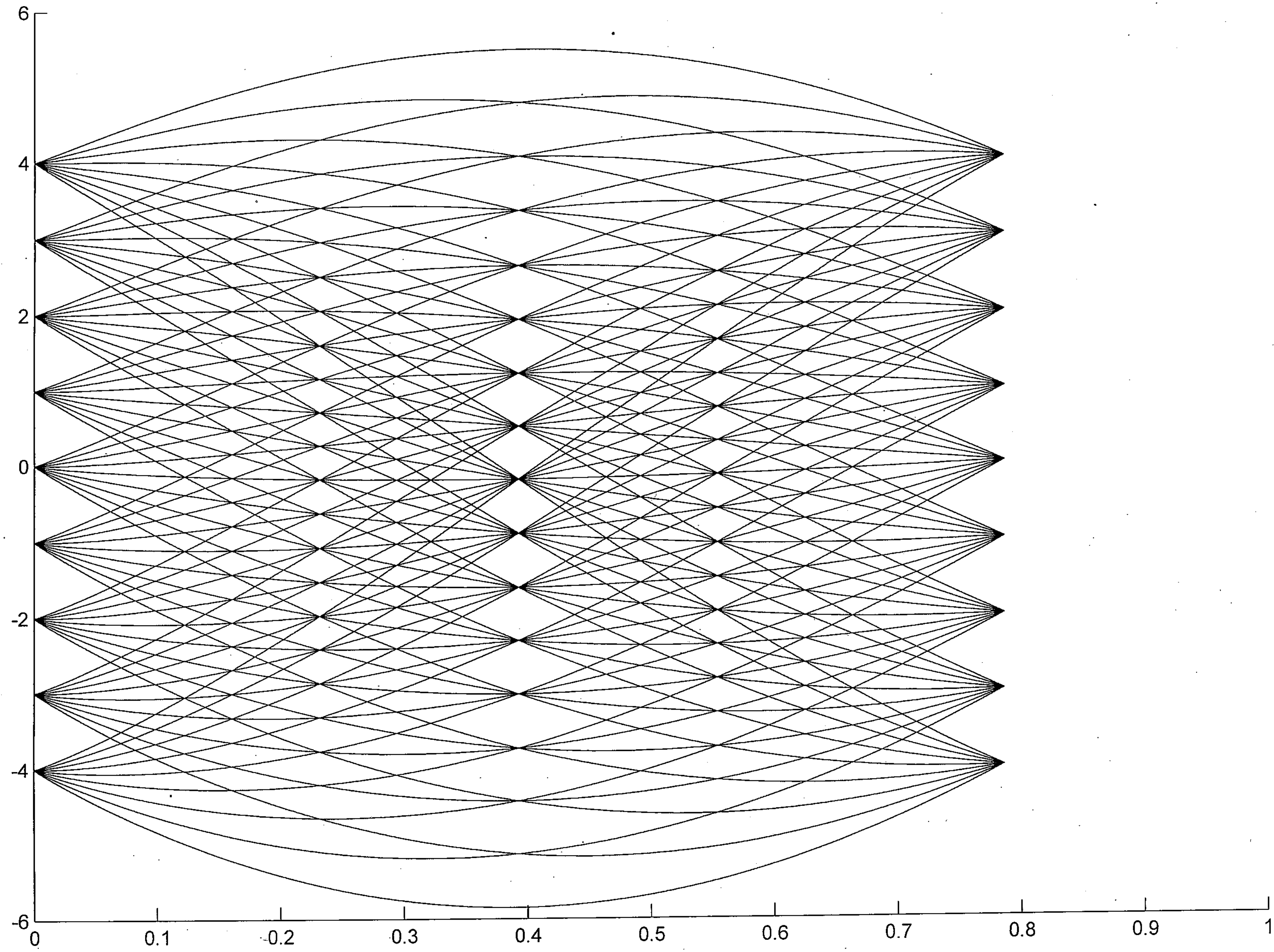
Set $y_P = u y_1 + v y_2$

$u = -\int \frac{y_2 r}{W}\, dx = \dots = -\frac{1}{2} x$.

& $v = \int \frac{y_1 r}{W}\, dx = \dots = \frac{1}{4} \ln|\sin 2x|$

$\Rightarrow \boxed{y_P = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \ln|\sin 2x|}$

& gen. sol. is $y = y_H + y_P$.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4.4 Summary of ODE techniques and types of equations you should know

- First order, directly integrable
- First order, separable
- First order, linear, integrating factor

revision.

- First order existence and uniqueness criteria
- First order, exact
- Second order homogeneous, linear, constant coefficients revision.
- Second order nonhomogeneous, constant coefficients, method of undetermined coefficients for certain cases
- Reduction of order, i.e. for $y'' + p(x)y' + q(x)y = 0$, if we have y_1 find y_2 by setting $y_2 = u(x)y_1$. revision.
- Second order nonhomogeneous, variation of parameters.

(Stewart pp 1158-1160)