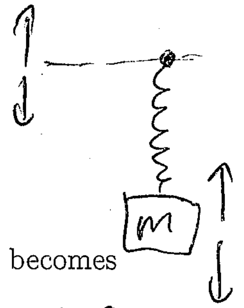
5 Forced oscillations - resonance, beats, practical resonance

By the end of this section, you should be able to answer the following questions:

- How to determine the steady state solution of a forced oscillator?
- What is resonance?
- How do beats arise?



Recall ODE for free oscillations with damping:

$$my'' + cy' + ky = 0.$$

Now if we have an external force $r(t)$ acting on the body, the equation becomes

$$my'' + cy' + ky = \boxed{r(t)}.$$

NII: my'' = net force = spring restoring force + damping force + $\boxed{\text{external driving force}}$

$r(t)$ is called the input or driving force.

Of particular interest are periodic inputs of the form

$$r(t) = F_0 \cos \omega t, \quad F_0 > 0, \; \omega > 0,$$

so that the ODE becomes

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{6}$$

We have already seen how to determine y_H.

To determine y_P, by the method of undetermined coefficients, we set

$$y_P = a \cos \omega t + b \sin \omega t. \quad *$$

After substituting into (6), also setting $\omega_0 = \sqrt{k/m}$, we obtain

try it!

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \tag{7}$$

$$b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \tag{8}$$

Note we need to modify our initial guess if $\omega = \omega_0$. & $c = 0$

We now look at the different cases when the system is damped ($c > 0$) or undamped $\boxed{(c = 0).}$ $\rightarrow y_H = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

5.1 $\boxed{\text{Undamped}}$ forced oscillations

$y_H = A\cos\omega_0 t + B\sin\omega_0 t$

In this case $\boxed{c = 0}$. Assume $\omega^2 \neq \omega_0^2$. Then

$$y_P = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

(trig. id.)

We can therefore write the general solution as

$$y(t) = \underbrace{C\cos(\omega_0 t - \delta)}_{y_H} + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

This represents a superposition of two harmonic oscillations. Their frequencies are the natural frequency $\omega_0/2\pi$ (cycles/sec) of the system and the frequency $\omega/2\pi$ of the input.

The maximum amplitude of y_P in this case is

$$a_0 = \frac{F_0}{k}\rho, \quad \rho = \frac{1}{1 - (\omega/\omega_0)^2},$$

where ρ is called the resonance factor. As $\omega \to \omega_0$, ρ and $a_0 \to \infty$. This phenomenon of excitation of large oscillations by matching input and natural frequencies ($\omega = \omega_0$) is known as resonance.

In the case of resonance the ODE can be written

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t.$$

$\omega = \omega_0$

The modified guess then gives

$$y_P = t(a \cos \omega_0 t + b \sin \omega_0 t).$$

Determining a and b by substitution into the ODE leads to

$$y_P = \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

These oscillations grow as t increases.

When we are close to resonance, beats arise.

ω close to ω_0

Take the solution

$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t)$$

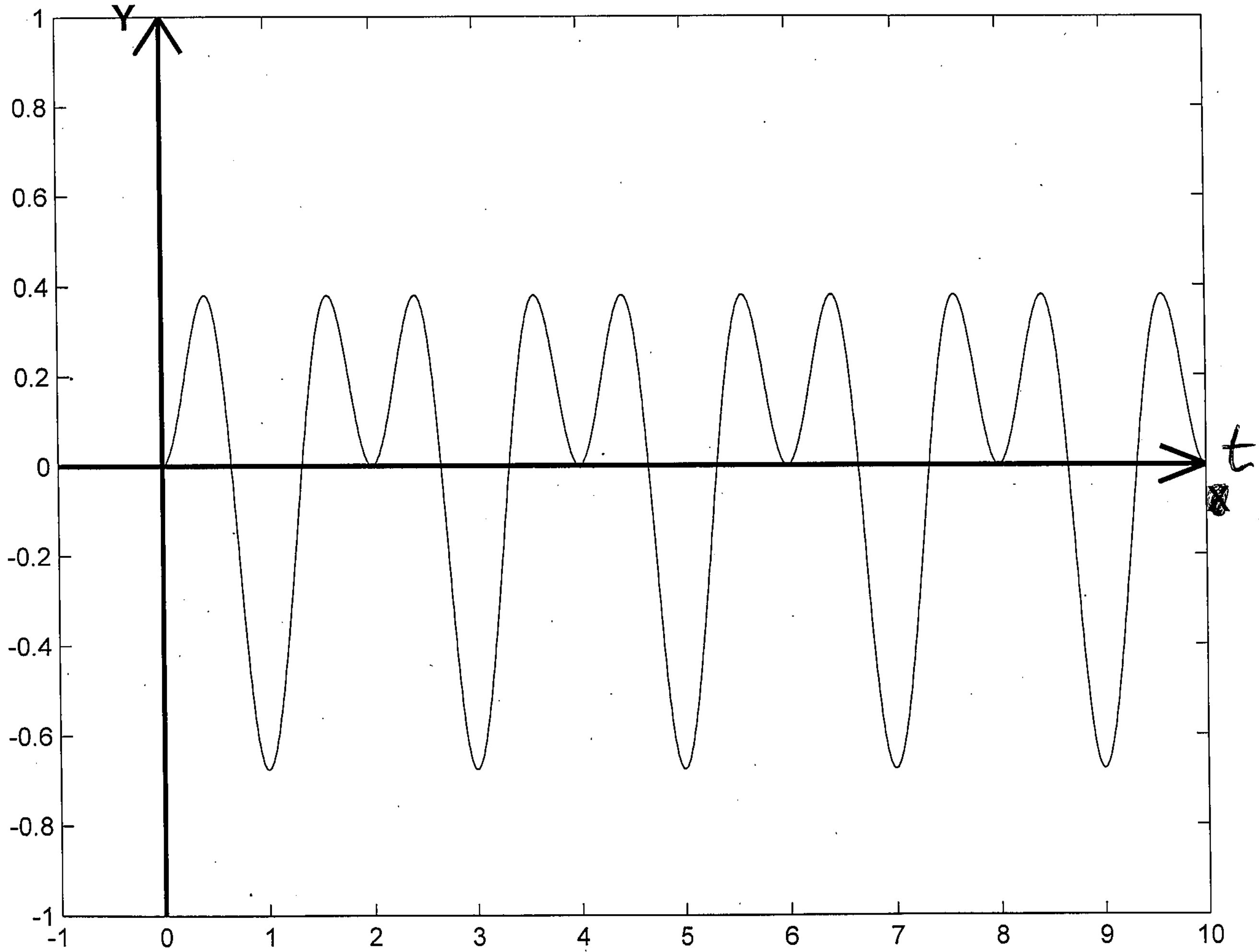
corresponding to the initial conditions $y(0) = 0$, $y'(0) = 0$. This can be rewritten

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right)$$

Since we are close to resonance, $\omega_0 - \omega$ is small, so the period of the last sine term is large, giving rise to beats.

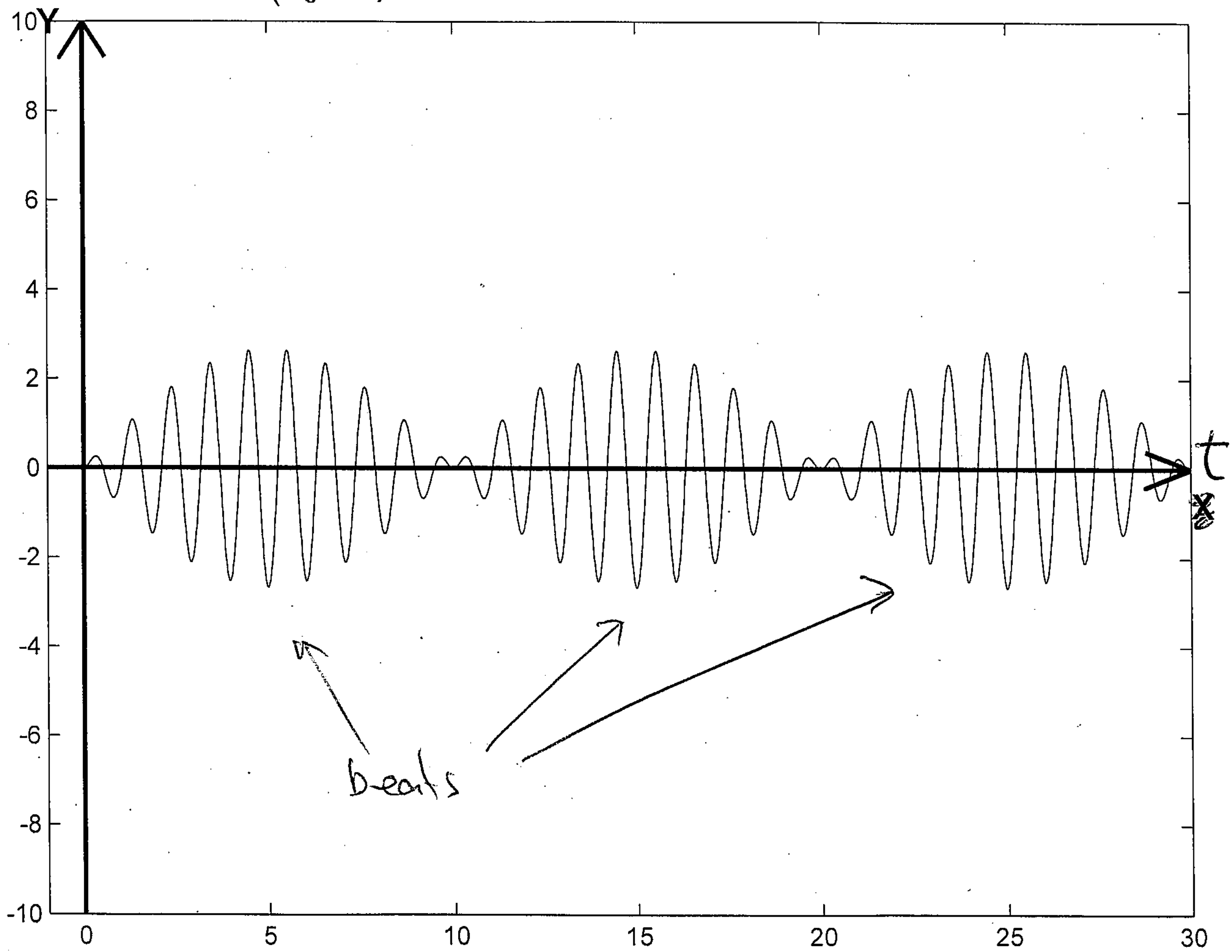
$\omega_0 = 2\pi, \ \omega = \pi, \ F_0 = 10, m = 1$

$$y = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right)$$



$$\omega_0 = 2\pi, \quad \omega = 1.8\pi, \quad F_0 = 10, \quad m = 1$$

$$y = \frac{2F_0}{m(\omega_0^2-\omega^2)} \sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$$



5.2 Damped forced oscillations

$\psi = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$

With damping, $c > 0$ and we know already that

$$y_H = e^{-\frac{c}{2m}t}(A\cos(\psi t) + B\sin(\psi t))$$

(remember underdamping gives damped oscillations).

$y_H \to 0$ as $t \to \infty$, so the general solution in the forced case will approach y_P as $t \to \infty$. That is, the general solution $\boxed{y(t) = y_H + y_P}$ is a transient solution and approaches a steady-state solution which is given by y_P.

This is what happens in practice, because no physical system is completely undamped.

With damping, the amplitude is finite as ω becomes close to ω_0, but may have a large maximum at some value of ω. In otherwords, some input may excite large destructive oscillations even with damping.

For the steady state solution, we have already seen that

$$\begin{aligned} y_P &= a\cos\omega t + b\sin\omega t \\ &= C^*\cos(\omega t - \eta) \end{aligned}$$

with a and b given by (7) and (8) respectively.

The amplitude C^* of y_P is given by

$$\begin{aligned} C^* &= \sqrt{a^2+b^2} \\ &= \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}} \end{aligned} \quad *$$

Treating the amplitude as a function of ω, $C^*(\omega)$ will have a maximum when

$$\frac{dC^*}{d\omega} = 0, \qquad \text{(check this)}$$

that is, when $c^2 - 2m^2(\omega_0^2 - \omega^2) = 0$, or when

$$\omega^2 = \omega_0^2 - \frac{c^2}{2m^2} \;= \frac{2\omega_0^2m^2 - c^2}{2m^2} \tag{9}$$

For sufficiently large damping, $c^2 > 2m^2\omega_0^2$, (9) has no real solutions, and C^* decreases in a monotone way as ω increases.

If $c^2 \leq 2mk$, (9) has one real solution (remember $\omega > 0$)

$$\omega = \omega\text{max} = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}}$$

and

$$C^*_{\max} = C^*(\omega_{\max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}.$$

This is what we call practical resonance.

The ratio C^*/F_0 is called the amplification, which $\to \infty$ as $c \to 0$ in agreement with the case of resonance.

Example O.D.E.: $y(0)=0, \ y'(0)=0$

$$y'' + y' + y = 10\cos(\omega t)$$

$$\Leftrightarrow \quad m = 1, \ c = 1, \ k = 1, \ F_0 = 10.$$

$$\omega_{\max} = \frac{1}{\sqrt{2}}$$

$$C^*_{\max} = \frac{20}{\sqrt{3}} \approx 11.547.$$

$$
\begin{aligned}
y(t) = \\
& -\frac{5e^{-\frac{1}{2}t}(1+\omega^2)}{(1-\omega^2)^2+\omega^2}\left(\frac{2(1-\omega^2)}{1+\omega^2}\cos\left(\frac{\sqrt{3}}{2}t\right)+\sin\left(\frac{\sqrt{3}}{2}t\right)\right) \\
& +\frac{10}{(1-\omega^2)^2+\omega^2}\Big((1-\omega^2)\cos(\omega t)+\omega\sin(\omega t)\Big)
\end{aligned}
$$