6 Hyperbolic functions

By the end of this section, you should be able to answer the following questions:

- What is the definition of the sinh and cosh functions?
- What is the definition of the inverse hyperbolic functions?
- What are the derivatives and anti-derivatives of these functions?
- How are hyperbolic functions used in the catenary problem? (reading)

6.1 Properties of hyperbolic functions

We define the functions $\cosh(x)$ and $\sinh(x)$ by

$$\begin{aligned} \cosh(x) &= \frac{e^x + e^{-x}}{2}, \\ \sinh(x) &= \frac{e^x - e^{-x}}{2}. \end{aligned}$$

We can check by direct calculation that

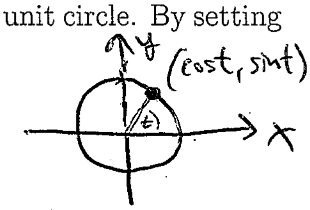
$$\cosh^2(x) - \sinh^2(x) = 1.$$

Compare this with the identity

$$\cos^2(\theta) + \sin^2(\theta) = 1 \tag{10}$$

for trig functions. The identity (10) allows us to parametrise a unit circle. By setting $x(t) = \cos(t)$, $y(t) = \sin(t)$, we have

$$\cos^2(t) + \sin^2(t) = x^2 + y^2 = 1,$$

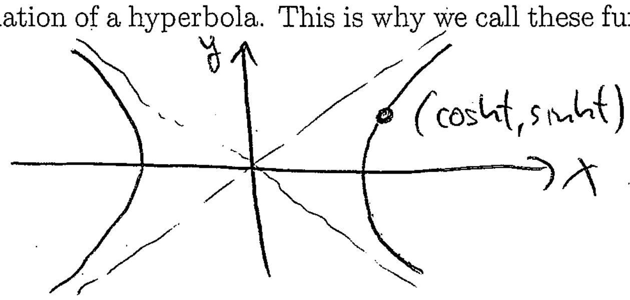


which is the equation of the unit circle.

If we set $x(t) = \cosh(t)$ and $y(t) = \sinh(t)$, this gives a parametrisation for a hyperbola (only the right branch), since

$$\cosh^2(t) - \sinh^2(t) = x^2 - y^2 = 1,$$

which is the equation of a hyperbola. This is why we call these functions "hyperbolic functions".



These hyperbolic functions satisfy properties similar to their trigonometric counterparts. For example

$$\begin{aligned}\frac{d}{dx}(\cosh(x)) &= \frac{e^x - e^{-x}}{2} = \sinh(x),\\ \frac{d}{dx}(\sinh(x)) &= \frac{e^x + e^{-x}}{2} = \cosh(x).\end{aligned}$$

$\cosh(0) = 1$, $\cosh(x) \geq 1$, $\cosh(x)$ is an even function. $\cosh(-x) = \cosh(x)$

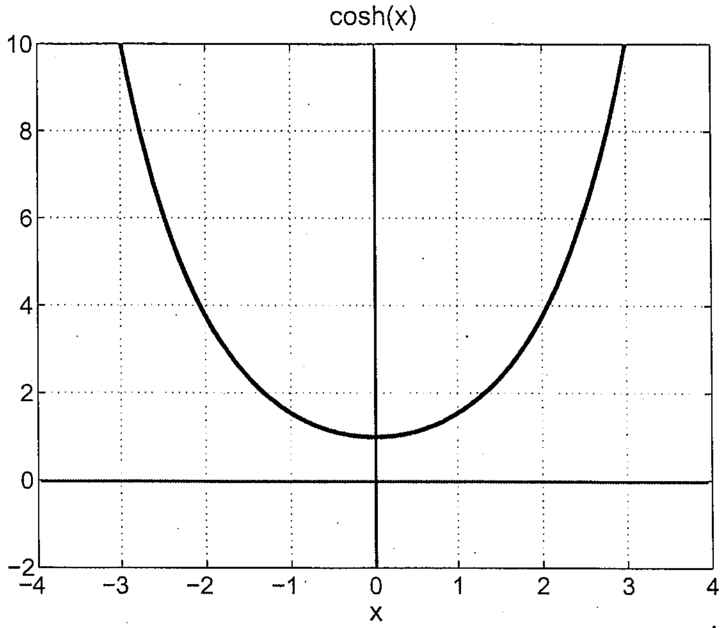
$\sinh(0) = 0$, $\sinh(x)$ is an odd function. $\sinh(-x) = -\sinh(x)$

We also define

$$\begin{aligned}\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{1 - e^{-2x}}{1 + e^{-2x}}, \quad |\tanh(x)| < 1,\\ \coth(x) &= \frac{\cosh(x)}{\sinh(x)}.\end{aligned}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)}$$



domain $\mathbb{R}$.
range $[1,\infty)$

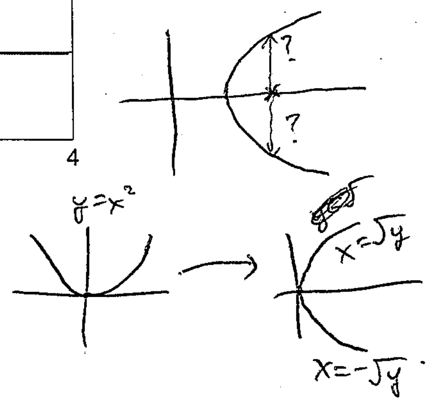
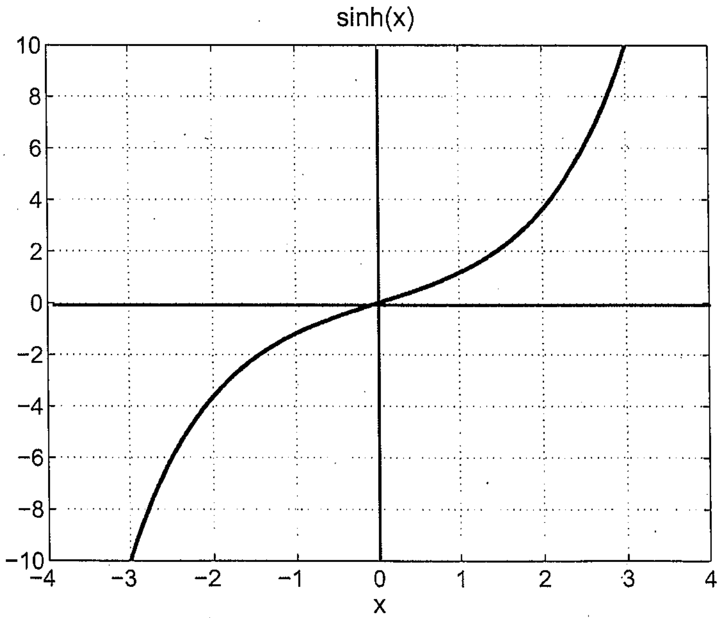
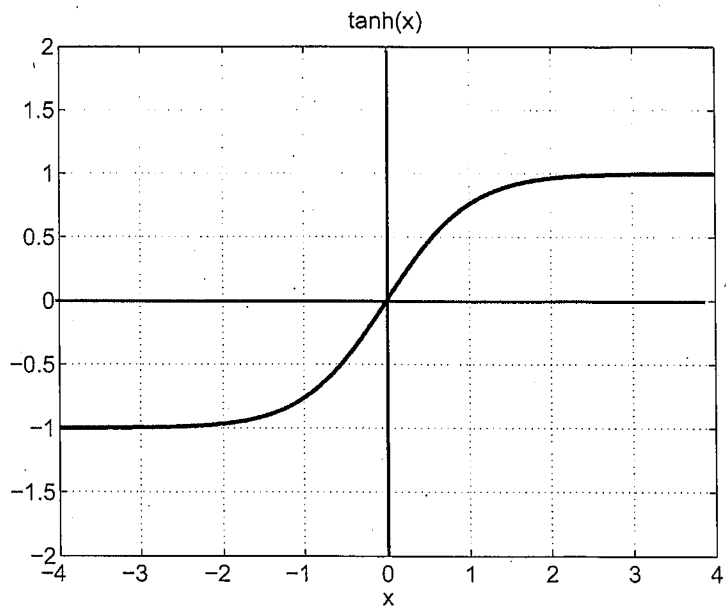


Figure 1: Graph of $\cosh(x)$



dom, range $\mathbb{R}$.

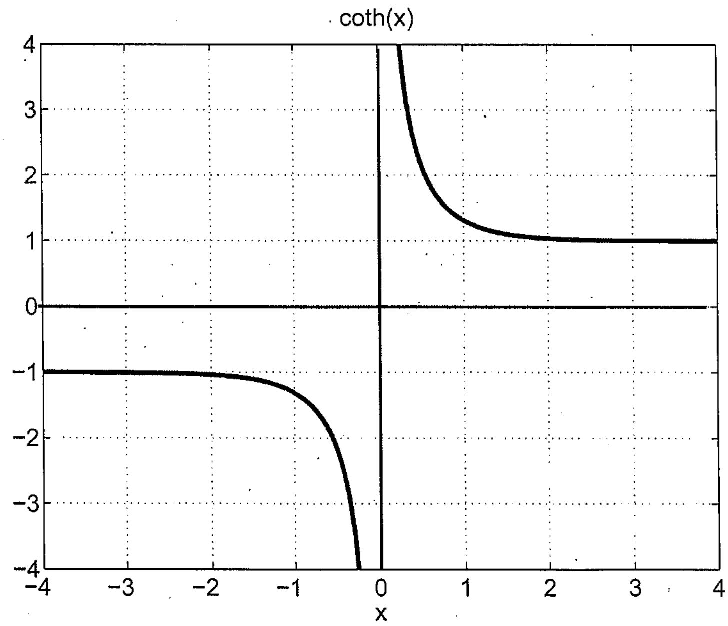
Figure 2: Graph of $\sinh(x)$



dom $\mathbb{R}$

range $(-1,1)$

Figure 3: Graph of $\tanh(x)$



dom

$(-\infty,0)\cup(0,\infty)$

or $\mathbb{R}\setminus\{0\}$.

range

$(-\infty,-1)\cup(1,\infty)$

or

$\mathbb{R}\setminus[-1,1]$.

Figure 4: Graph of $\coth(x)$

6.2 Inverse hyperbolic functions

$\left(\neq \frac{1}{\cosh x} \leftarrow \text{this is } \operatorname{sech} x\right)$

The inverse function of cosh is denoted arcosh. $\equiv \cosh^{-1}$

The inverse function of sinh is denoted arsinh.

The inverse function of tanh is denoted artanh.

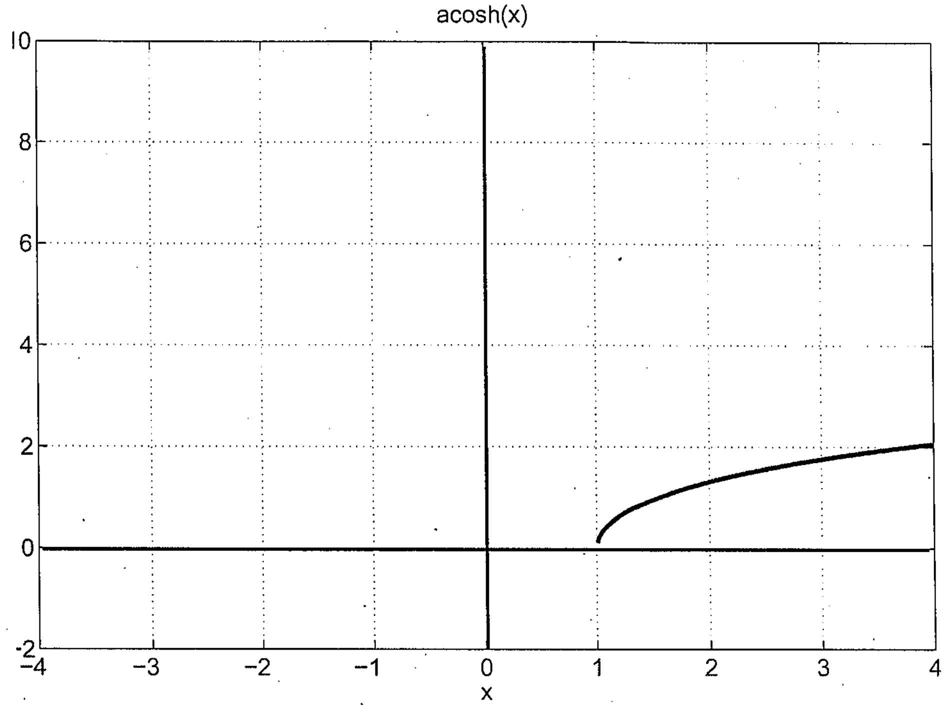


Figure 5: Graph of arcosh (x)

– restrict domain of cosh to $[0, \infty)$

$\Longrightarrow$ range of $\cosh^{-1}$ is $[0, \infty)$

& domain of $\cosh^{-1}$ is $[1, \infty)$

sinh^{-1} dom & range $\mathbb{R}$.

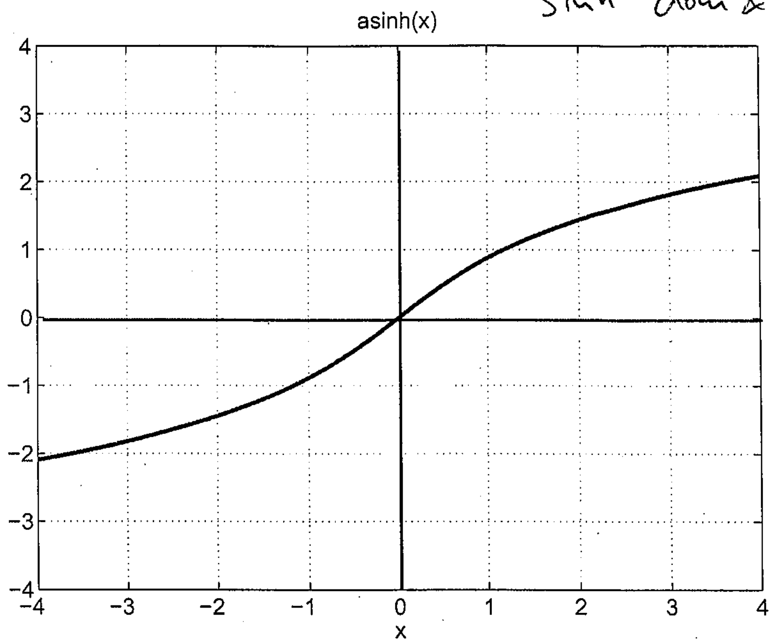
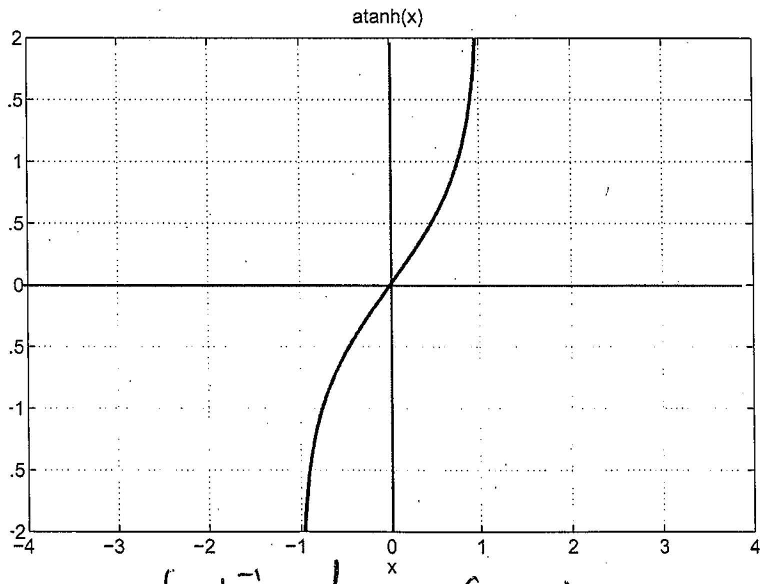


Figure 6: Graph of arsinh (x)



tanh^{-1} dom $(-1,1)$, range $\mathbb{R}$.

Figure 7: Graph of artanh (x)

We have the following:

$$\begin{aligned}\int \frac{dx}{\sqrt{1+x^2}} &= \operatorname{arsinh}(x) + c \\ \int \frac{dx}{\sqrt{x^2-1}} &= \operatorname{arcosh}(x) + c,\ x > 1.\end{aligned}$$

6.2.1 Show that $\frac{d}{dx}(\operatorname{arsinh}(x)) = \frac{1}{\sqrt{1+x^2}}$

$y = \sinh^{-1} x \implies \boxed{x = \sinh y}$

differentiate both sides w.r.t. x

$\implies \frac{d}{dx}(x) = \frac{d}{dx} \sinh y$

$\implies 1 = \frac{dy}{dx} \cdot \cosh y$ (chain rule)

$\implies \boxed{\frac{dy}{dx} = \frac{1}{\cosh y}}$

but $\cosh^2 y - \sinh^2 y = 1$

$\implies \cosh y = \sqrt{1 + \sinh^2 y}$ (≥ 1)

$= \sqrt{1 + x^2}$

$\implies \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$ — result.

6.2.2 Evaluate the integrals $\int \frac{dx}{\sqrt{1+x^2}}$ (a) **and** $\int \frac{dx}{\sqrt{x^2-1}}$ (b)

(a) set $x = \sinh t \Rightarrow$ in $\int$, $dx = \cosh t \, dt$.

$$\Rightarrow \int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh t \, dt}{\sqrt{1+\sinh^2 t}}$$

$$= \int \frac{\cosh t}{\sqrt{\cosh^2 t}} \, dt$$

$$= \int dt = t + c = \sinh^{-1} x + c$$

(b) (assume $x > 1$, always be mindful of domain & range in sub's)

Set $\boxed{x = \cosh t} \Rightarrow$ in $\int$, $dx = \sinh t \, dt$.

$$\Rightarrow \int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sinh t \, dt}{\sqrt{\cosh^2 t - 1}} = \int \frac{\sinh t \, dt}{\sqrt{\sinh^2 t}}$$

$$= \int dt = t + c = \cosh^{-1} x + c$$

(for $x > 1$)

Caution: Convention $\sqrt{\ }$ always ≥ 0

$\sqrt{\sinh^2 t} = \sinh t$? only if $t \geq 0$

for $t < 0$, $\sinh t < 0$

$\Rightarrow \sqrt{\sinh^2 t} = -\sinh t$

6.2.3 Show that $\frac{d}{dx}(\text{artanh}(x)) = \frac{1}{1-x^2}, \quad |x| < 1$

Similar to p43.
(exercise)
~~cosh~~

Consider $\frac{\cosh^2 t}{\cosh^2 t} - \frac{\sinh^2 t}{\cosh^2 t} = \frac{1}{\cosh^2 t}$

$\Rightarrow \boxed{1 - \tanh^2 t = \text{sech}^2 t}$

Using partial fractions, we can also evaluate the integral

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + C.$$

Note: $1-x^2 = (1+x)(1-x)$

In fact, we have the following identities

$$\begin{aligned}
\text{artanh}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \\
\text{arsinh}(x) &= \ln\left(x + \sqrt{x^2+1}\right), \\
\text{arcosh}(x) &= \ln\left(x + \sqrt{x^2-1}\right).
\end{aligned}$$

6.2.4 **Show that** $\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

$y = \sinh^{-1} x \Rightarrow x = \sinh y.$

$\Rightarrow x = \dfrac{e^y - e^{-y}}{2}$

$\Rightarrow 2x = e^y - e^{-y}$

(mult. by e^y) $\Rightarrow (e^y)^2 - 2xe^y - 1 = 0$

quadratic in e^y.

$\Rightarrow e^y = \dfrac{2x \pm \sqrt{4x^2 - 4.1.(-1)}}{2}$

$= x \pm \sqrt{x^2+1}$

Note $e^y > 0$, but $x - \sqrt{x^2+1} < 0$

$\Rightarrow$ must have

$e^y = x + \sqrt{x^2+1}$

$\Rightarrow \ln(e^y) = y = \ln\left(x + \sqrt{x^2+1}\right)$

MORE FUN ...

Euler's formula: $e^{\pm ix} = \cos x \pm i \sin x$

$\Rightarrow \cosh(ix) = \dfrac{e^{ix} + e^{-ix}}{2} = \cos x.$ $(x \in \mathbb{R})$

Also $\sinh(ix) = i \sin x.$