7 Introduction to double integrals, volume below a surface

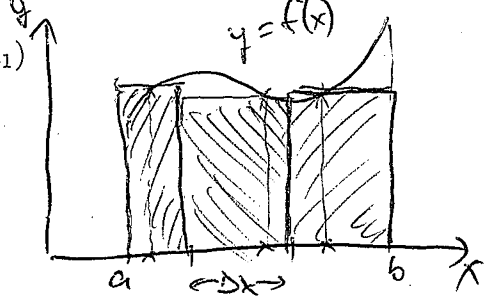
By the end of this section, you should be able to answer the following questions:

- What is the definition of volume below a surface? $z = f(x,y) > 0$
- What is the definition of a double integral?
- How are the two related?
- iterated integrals.

Recall that if $y = f(x)$, the area under the curve over the interval $I = [a, b]$ is

$$\boxed{\int_I f(x)dx} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

where $x_i^* \in [x_i, x_{i-1}]$.



7.1 Double integrals

Suppose we have a surface $z = f(x, y)$ above a planar region R in the x-y plane.

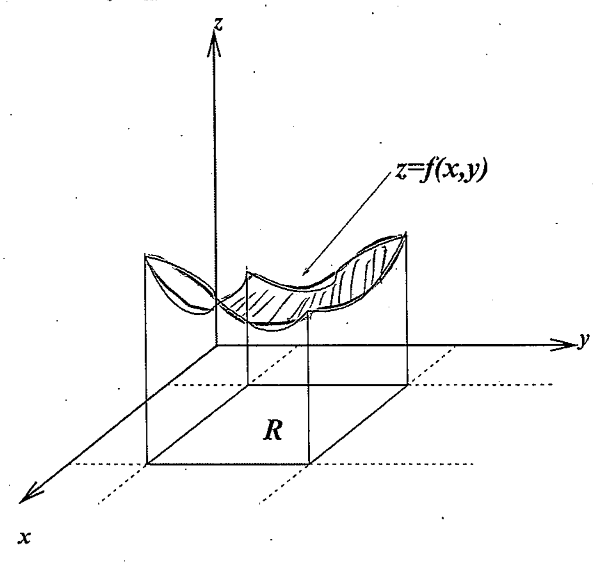
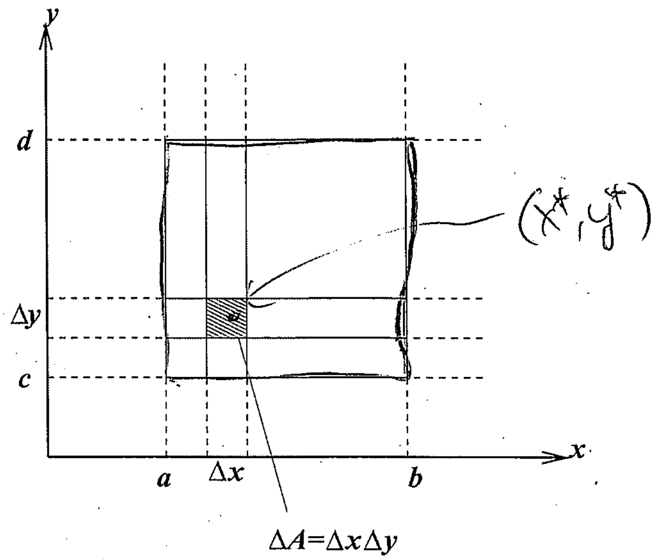


Figure 10: What is the volume V under the surface?

Before moving onto general regions, we start by considering the case where R is a rectangle. That is,

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ c \leq y \leq d\}.$$



Start by dividing R into subrectangles by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$, each of width $\Delta x = \dfrac{b-a}{m}$ and $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \dfrac{d-c}{n}$.

Combining these gives a rectangular grid R_{ij} with subrectangles each of area $\Delta A = \Delta x \Delta y$.

In each subrectangle take any point P_{ij} with co-ordinates (x^*_{ij}, y^*_{ij}).

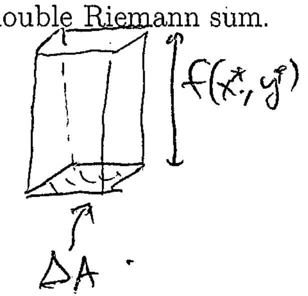
The volume of the box with base the rectangle ΔA and height the value of the function $f(x, y)$ at the point P_{ij} (so the box touches the surface at a point directly above P_{ij} - see figure 11) is

$$* \quad V_{ij} = \overbrace{f(x^*_{ij}, y^*_{ij})}^{\text{height}} \underbrace{\Delta A.}_{\text{ex. base area.}}$$

Then for all the subrectangles we have an approximation to the required volume V:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A,$$

the double Riemann sum.



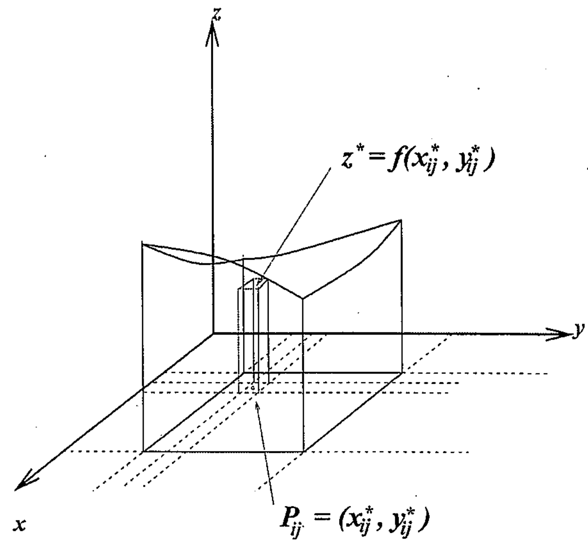


Figure 11: The rectangular box whose volume is $z^*\Delta A$.

Let $\Delta x \to 0$ and $\Delta y \to 0$, ie $m \to \infty$ and $n \to \infty$, then we *define* the volume to be

$$V = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A,$$

$\Delta A \to 0$

if the limits exist and we write this as

$$\boxed{\iint_R f(x,y)dA.}$$

$\leftarrow$ vol.

We call f integrable if the limits exist. Note that every continuous function is integrable.

7.2 Properties of the double integral

(i) $\displaystyle\iint_R (f \pm g)dA = \iint_R f dA \pm \iint_R g dA$

$f(x,y)$, $g(x,y)$

(ii) $\displaystyle\iint_R c f dA = c \iint_R f dA$

c const.

$R = R_1 \cup R_2$

(iii) $\displaystyle\iint_R f dA = \iint_{R_1} f dA + \iint_{R_2} f dA$

(iv) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$ then

$$\iint_R f dA \geq \iint_R g dA$$

7.3 Iterated integrals

We define $\int_c^d f(x,y)dy$ to mean that x is fixed and $f(x,y)$ is integrated with respect to y from $y=c$ to $y=d$. So

$$\boxed{A(x) = \int_c^d f(x,y)dy}$$

is a function of x only.

If we now integrate $A(x)$ with respect to x from $x=a$ to $x=b$ we have

$$\begin{aligned}
\int_a^b A(x)dx &= \int_a^b \left[\int_c^d f(x,y)dy\right] dx \\
&= \int_a^b \left(\int_c^d f(x,y)\ dy\right) dx
\end{aligned}$$

This is called an iterated integral.

7.3.1 Example: evaluate $\int_0^2 \int_1^3 x^2 y\ dy\ dx$

$$= \int_0^2 \left(\int_1^3 x^2 y\ dy\right) dx$$

$$= \int_0^2 \left[\tfrac{1}{2} x^2 y^2\right]_{y=1}^{y=3} dx$$

$$= \int_0^2 \left(\tfrac{9}{2} x^2 - \tfrac{1}{2} x^2\right) dx.$$

$$= \int_0^2 4x^2\, dx = \frac{4}{3} x^3 \Big|_0^2 = \frac{4}{3} \times 2^3 = \frac{32}{3}.$$

Now try integrating the other way around:

7.3.2 Example: evaluate $\int_1^3 \int_0^2 x^2 y \, dx \, dy$

$$= \int_1^3 \left(\int_0^2 x^2 y \, dx \right) dy.$$

$$= \int_1^3 \left[\frac{1}{3} x^3 y \right]_{x=0}^{x=2} dy$$

$$= \int_1^3 \left(\frac{8}{3} y - 0 \right) dy.$$

$$= \frac{4}{3} y^2 \Big|_1^3 = \frac{4}{3}\left(3^2 - 1^2\right) = \frac{4 \times 8}{3} = \frac{32}{3},$$

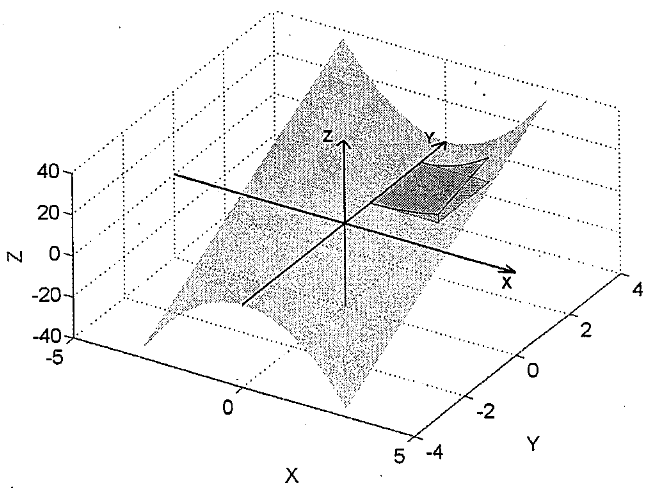


Figure 12: We have just calculated the volume of the solid outlined above.