8 Fubini's theorem, volume by slabs

By the end of this section, you should be able to answer the following questions:

- What is Fubini's theorem?
- How is the double integral related to the iterated integral?
- How do you estimate the volume below a surface using slabs?

8.1 Fubini's theorem

If $f(x,y)$ is integrable on the rectangle

$$R = \{(x,y)|a \le x \le b,\ c \le y \le d\},$$

then

$$\begin{aligned}\iint_R f(x,y)dA &= \int_a^b \left(\int_c^d f(x,y)\ dy\right) dx \\ &= \int_c^d \left(\int_a^b f(x,y)\ dx\right) dy\end{aligned}$$

"dx dy" or "dy dx".

8.2 Example: evaluate $\iint_R (x^2+y^2)dA$ where $R = \{(x,y)|0 \le x \le 2,\ \underline{0 \le y \le 1}\}$

By Fubini's theorem.

$$\iint_R (x^2+y^2)\,dA = \int_0^2\left(\int_0^1 (x^2+y^2)\,dy\right)dx$$

$$= \int_0^2 \left[x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=1} dx.$$

$$= \int_0^2 \left(x^2 + \frac{1}{3} - 0\right) dx.$$

$$= \frac{1}{3}x^3 + \frac{1}{3}x \Big|_0^2 = \frac{8}{3} + \frac{2}{3} = \frac{10}{3}.$$

Exercise, try

$$\int_0^1\int_0^2 (x^2+y^2)\,dx\,dy$$

$\longrightarrow$ same result.

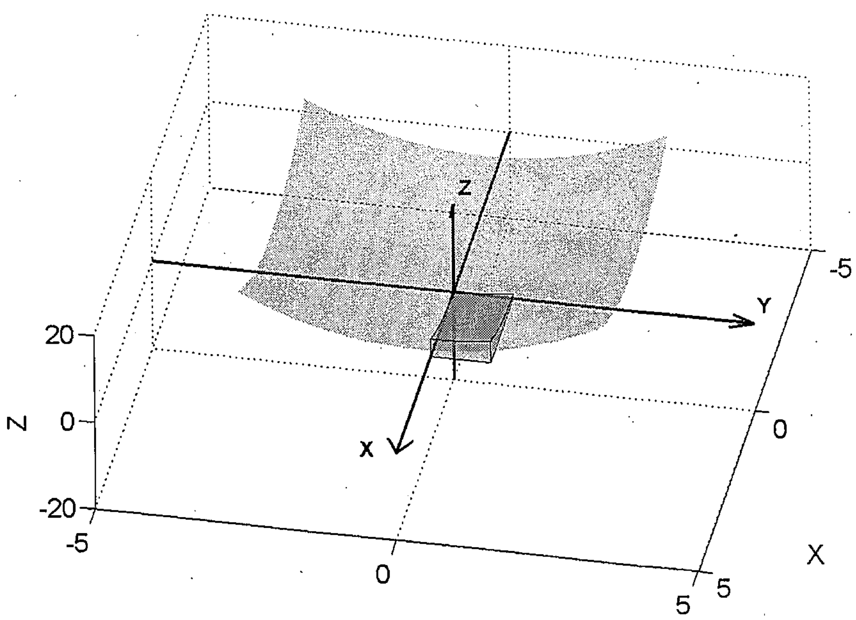


Figure 13: A representation of the volume in example 8.2.

8.3 Interpreting Fubini's theorem in terms of volume

Fubini's theorem is the key result that tells us how to evaluate a double integral. We can see the relation between the iterated integral and the double integral by considering an alternative way of calculating the volume below a surface.

Suppose we want to find the volume below the surace $\boxed{z = x^2y}$ above the square region $0 \leq x \leq 8$ and $0 \leq y \leq 4$.

A natural way to solve this problem is to break the region up into slabs of equal depth $\Delta y = y_{j+1} - y_j$ located at y_j, and add up the volume of the slabs

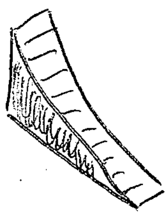
$$V \approx \sum_j \Delta V,$$

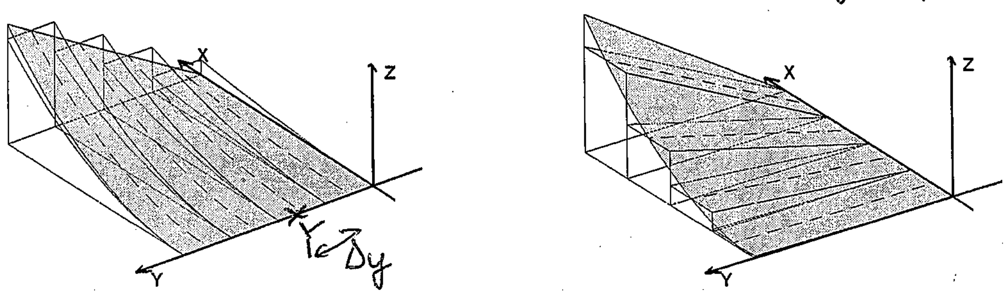
where ΔV the volume of the jth slab. Figure 14 below shows two ways of doing this using four slabs in each case. The ~~top~~ left diagram follows the method outlined here, taking slabs of thickness Δy.

surface $z = f(x,y)$

4 slabs parallel to x-z plane

4 slabs parallel to y-z plane.





Top of slab $f(x,Y)$, same cross-sectional are

Figure 14: Two ways of approximating the volume under $z = x^2y$ using four slabs.

If the slab is very thin (i.e. $\Delta y \ll 1$) then the volume of each slab is

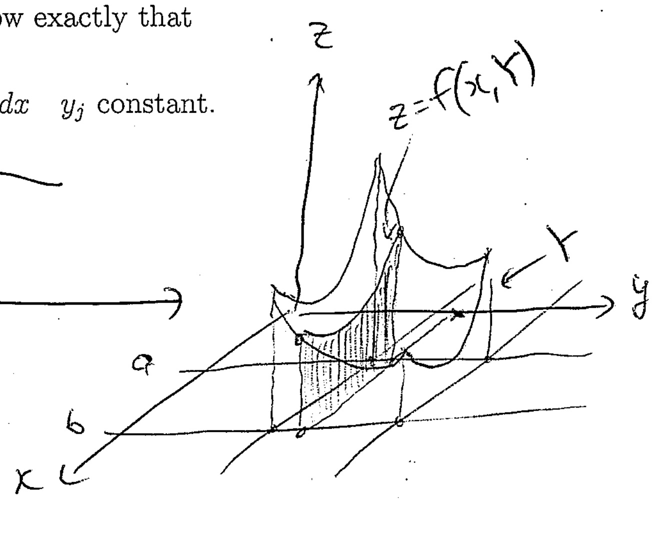
$$\Delta V \approx \underline{\text{Area of slab}} \times \text{Depth} = C(y_j)\Delta y.$$

(cross-sectional area)

Here $C(y_j)$ is the area of the slab at the location y_j (and the result will depend on y_j!). From one-dimensional calculus we know exactly that

$$C(y_j) = \int_0^8 f(x, y_j)dx \quad y_j \text{ constant.}$$

area of shaded region



It is easy to compute this as a regular integral since y_j does not vary with x. Putting all this together

$$V \approx \sum_j \Delta V_j \approx \sum_j C(y_j)\Delta y.$$

As the slabs become thinner and thinner ($\Delta y \to 0$) the approximation becomes more accurate and we can replace the summation by an integral[1]

$$V = \int_0^4 C(y)dy = \int_0^4 \left(\int_0^8 f(x,y)dx \right) dy$$

Note that the y is held constant in the inner integral.

A similar argument can be applied by considering slabs of depth Δx, located at x_j. In other words, take slabs that are parallel to the y-z plane.

p990 Stewart.

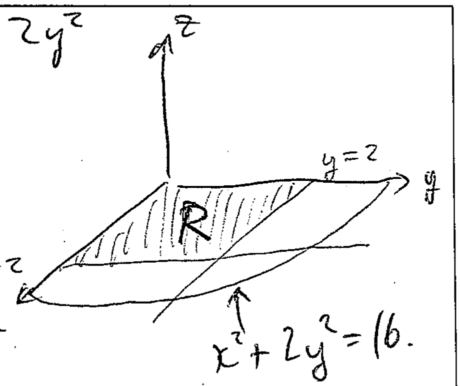
8.4 Example: find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes. $x=0,\ y=0,\ \boxed{z=0}$

x-y plane.

surface $z = 16 - x^2 - 2y^2$

$z = 0 = 16 - x^2 - 2y^2$

vol. above shaded region, below surface $= \iint_R (16 - x^2 - 2y^2)\,dA$



$R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2\}$.

$\Rightarrow$ vol. $= \int_0^2 \left(\int_0^2 (16 - x^2 - 2y^2)\,dx \right) dy$

$= \int_0^2 \left[16x - \frac{1}{3}x^3 - 2xy^2 \right]_{x=0}^{x=2} dy$

[1] Recall that is in fact the definition of an integral

$$= \int_0^2 \left(32 - \frac{8}{3} - 4y^2\right) dy$$

$$= \ldots = 48.$$

(try other order of integration)

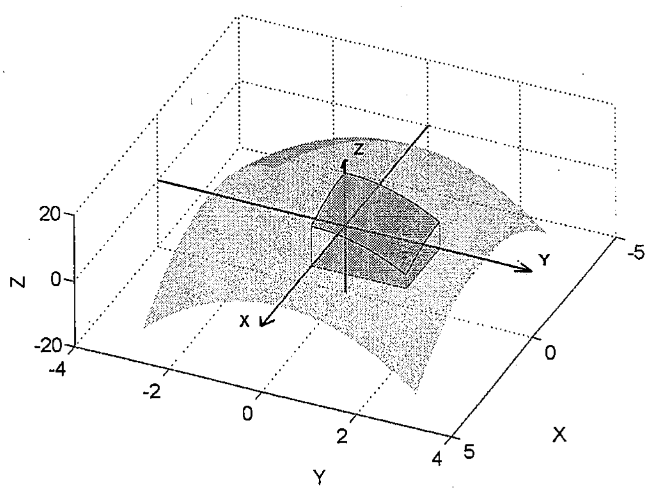


Figure 15: The volume of the solid of example 8.4 is below the surface $z = 16 - x^2 - 2y^2$ and above the x-y plane as shown.

8.5 Special case when $f(x,y) = g(x)h(y)$.

In this case we can separate the integral as follows.

rectangle R
⟹ bounds of
Integration
constant.

$$\iint_R f(x,y)dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy$$
$$= \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

x $\quad$ y

8.5.1 Example: $\iint_R \sin x \cos y \; dA$ where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

$(R = \{(x,y) \mid 0 \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq \frac{\pi}{2}\})$

$$\Rightarrow \iint_R \sin x \cos y \, dA = \int_0^{\pi/2}\int_0^{\pi/2} \sin x \cos y \, dx\, dy$$
$$= \left(\int_0^{\pi/2} \sin x \, dx\right)\left(\int_0^{\pi/2} \cos y \, dy\right)$$
$$= 1 \times 1 = 1$$

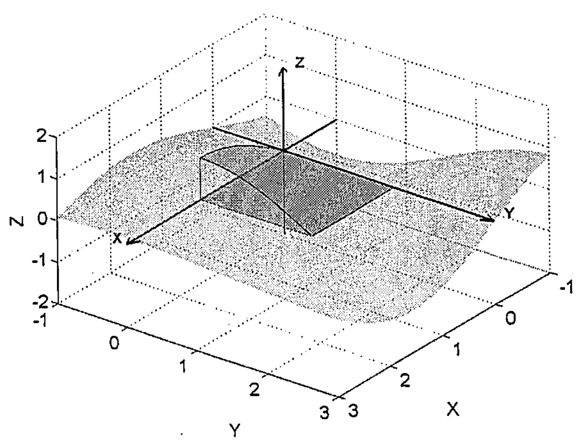


Figure 16: The volume calculated in example 8.5.1 is outlined above.