Stewart Chap. 16

9 Integrals over general regions

By the end of this section, you should be able to answer the following questions:

- How can you identify type I and II regions?
- How do you evaluate a double integral over type I and II regions?
- How can you evaluate a double integral over a more general region comprising finitely many type I and II regions?
- What is meant by net volume below a surface?

To find the double integral over a general region D instead of just a rectangle we consider a rectangle which encloses D and define

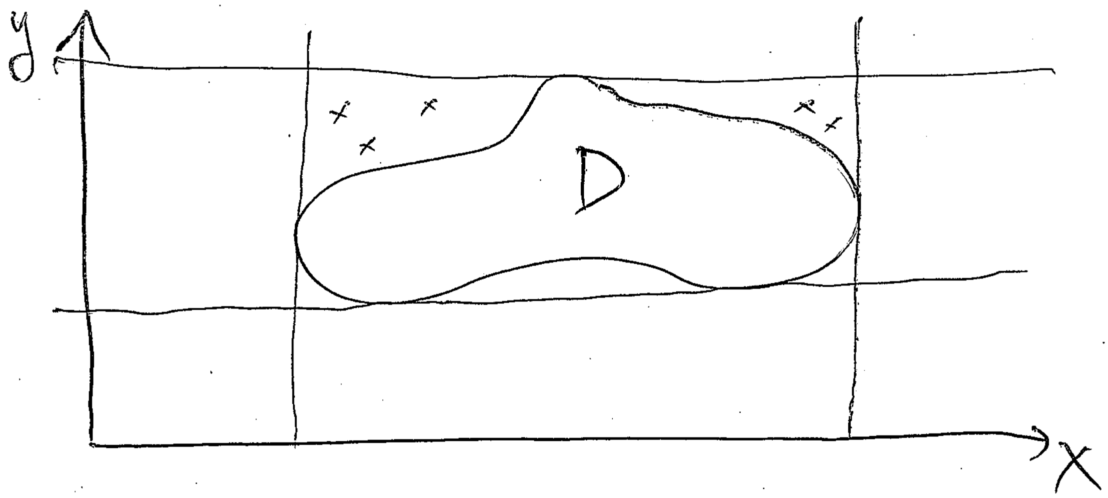
$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D \\ 0, & \text{if } (x,y) \in R \text{ but } \notin D \end{cases}$$

formal connection with sections 7, 8.

then

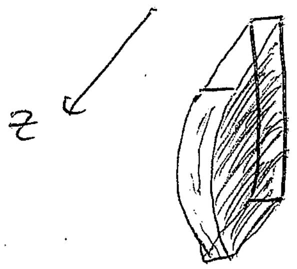
$$\iint_D f(x,y)dA = \iint_R F(x,y)dA$$

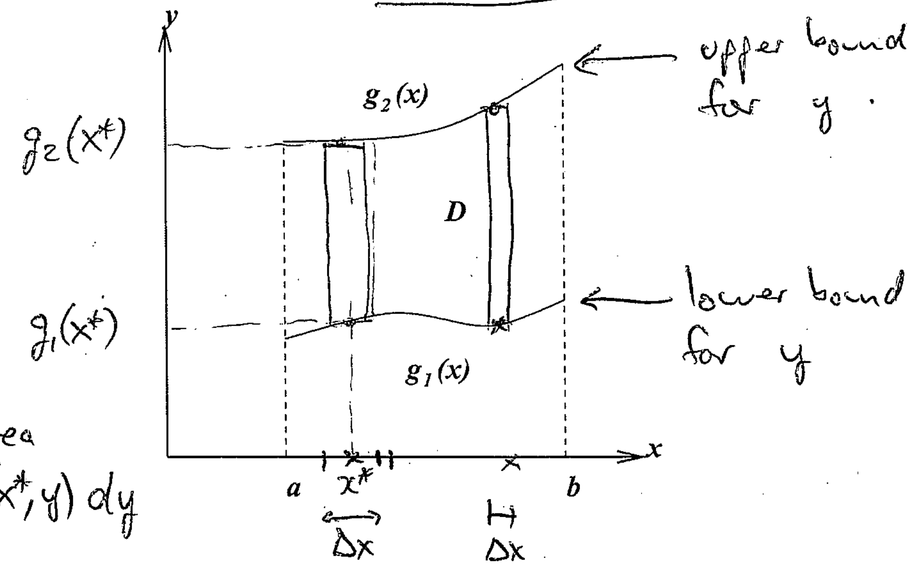
and we can proceed as before. It is possible to show that F is integrable if the boundary of D is bounded by a finite number of smooth curves of finite length. Note that F may still be discontinuous at the boundary of D.



9.1 Type I regions

A plane region D is of type I if it lies between the graph of two continuous functions of x. That is $\boxed{D = \{(x, y) | a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}.}$



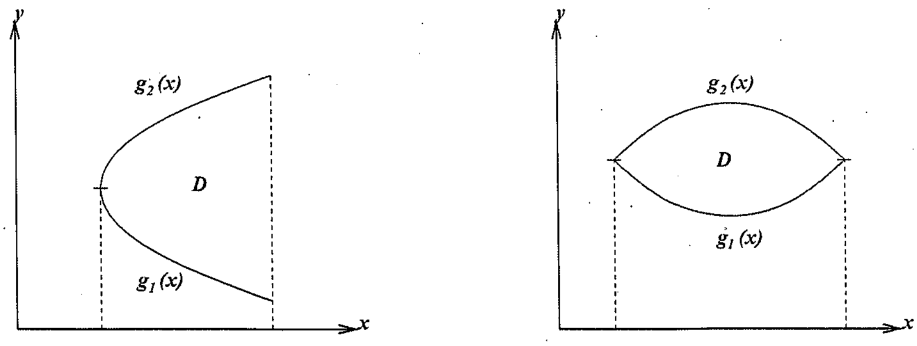


cross-sectional area of slab $= \displaystyle\int_{g_1(x^*)}^{g_2(x^*)} f(x^*, y)\, dy$

Figure 17: Type I regions are generally bounded by two constant values of x and two functions of x.

In practice, to evaluate $\iint_D f(x, y) dA$ where D is a region of type I we have

$$\iint_D f(x, y) dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\ dy \right) dx.$$



(a) (b)

Figure 18: Some more examples of type I regions.

9.1.1 Example: find $\iint\limits_D (4x+10y)dA$ where D is the region between the parabola $y = x^2$ and the line $y = x + 2$.

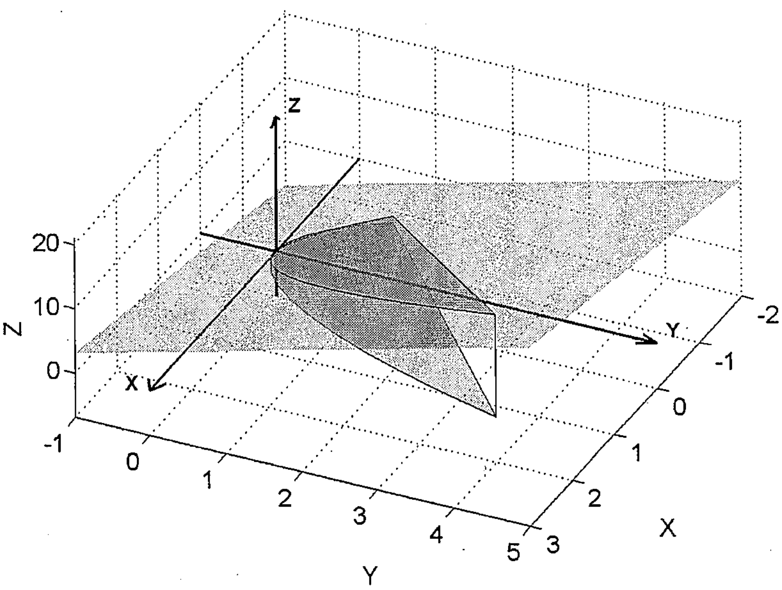
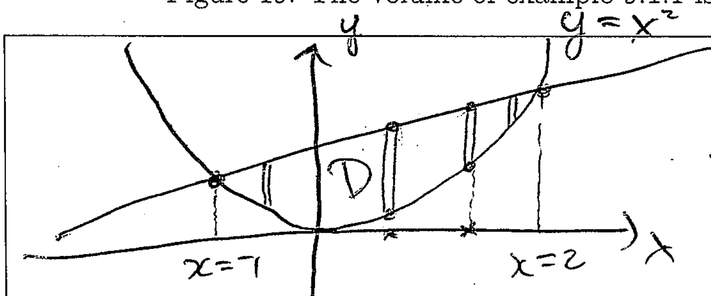


Figure 19: The volume of example 9.1.1 is outlined above.



$y=x+2$

points of intersection

$x^2 = x+2$

$\Rightarrow x=2,-1$

$D = \{(x,y) \mid -1 \le x \le 2,\ x^2 \le y \le x+2\}$

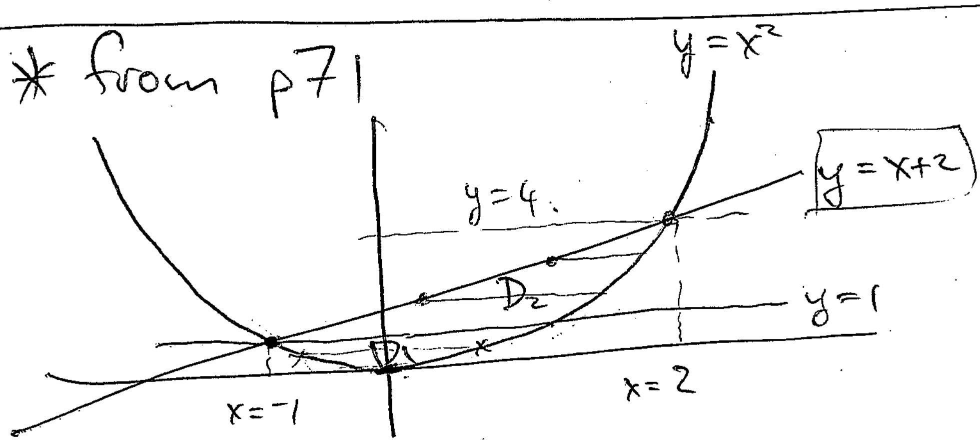
$$\Rightarrow \iint\limits_D (4x+10y)\,dA = \int_{-1}^{2}\left(\int_{x^2}^{x+2}(4x+10y)\,dy\right)dx$$

$$= \int_{-1}^{2} \Big[4xy + 5y^2 \Big]_{y=x^2}^{y=x+2} dx$$

$$= \int_{-1}^{2} \Big(4x(x+2) + 5(x+2)^2 - (4x^3 + 5x^4) \Big) dx$$

$$= \dots = 81$$

* from p71



Type $\underline{\text{II}}$?

$$D_1 = \{(x,y) \mid 0 \le y \le 1,\ -\sqrt{y} \le x \le \sqrt{y}\}.$$

$$D_2 = \{(x,y) \mid 1 \le y \le 4,\ y-2 \le x \le \sqrt{y}\}.$$

$$\iint_D = \iint_{D_1} + \iint_{D_2}$$

9.2 Type II regions

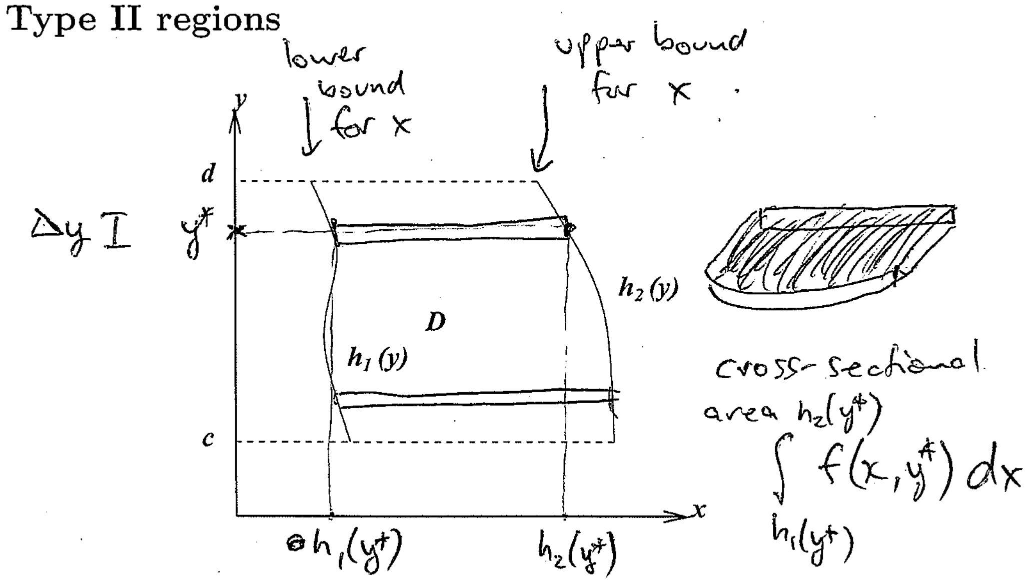


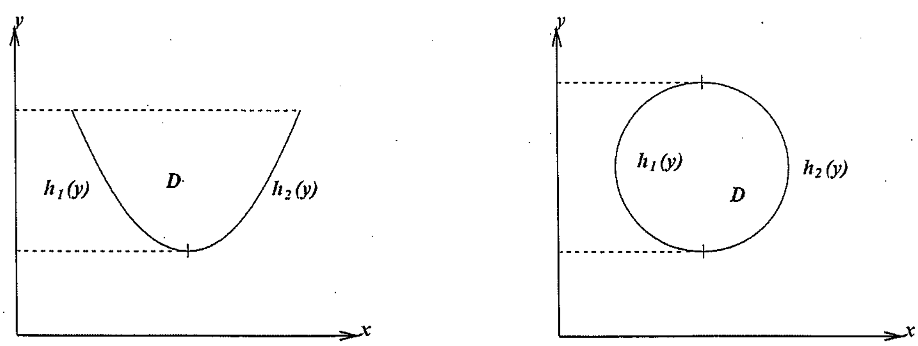
Figure 20: Type II regions are generally bounded by two constant values of y and two functions of y.

A plane region is of type II if it can be expressed by

$$\boxed{D = \{(x, y)|c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}.}$$

In practice, to evaluate $\iint\limits_D f(x, y)dA$ where D is a region of type II we have

$$\iint\limits_D f(x, y)dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y)\ dx \right) dy.$$



(a) (b)

Figure 21: Some more examples of type II regions.

9.2.1 Example: evaluate $\iint\limits_D xy\,dA$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

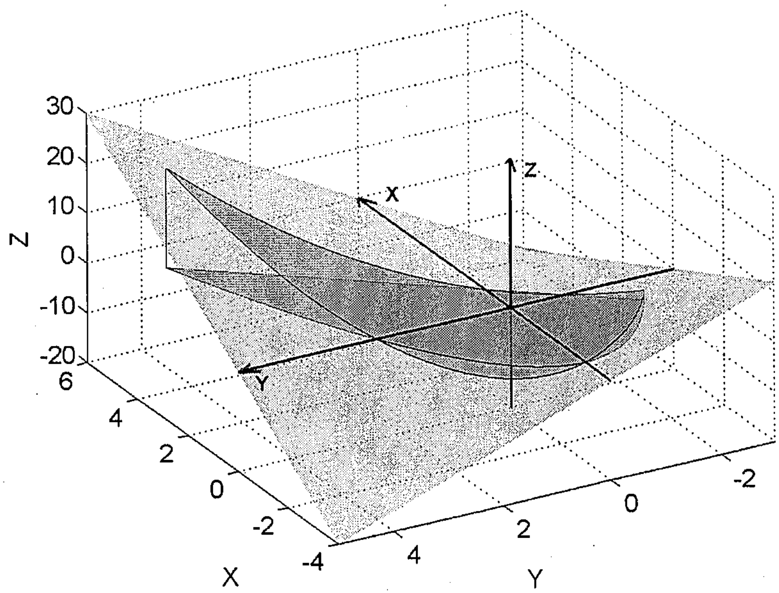


Figure 22: The volume of example 9.2.1 is outlined above. Note carefully that the surface is *above* the x-y plane only in the quadrants where $x, y > 0$ and $x, y < 0$. For x and y values in the other two quadrants, the surface is *below* the x-y plane. Hence in this example we are calculating the "net volume" lying above the x-y plane.

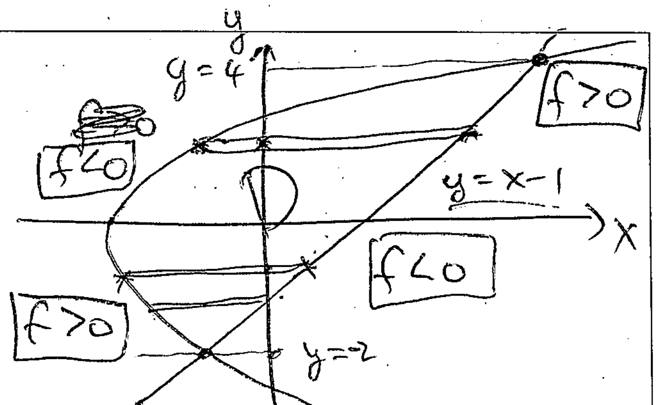
$f(x,y) = xy$

points of intersection

$y^2 = 2(y+1)+6$

$\Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow y = 4, -2$



$D = \{(x,y) \mid -2 \le y \le 4,\ \frac{y^2-6}{2} \le x \le y+1\}$

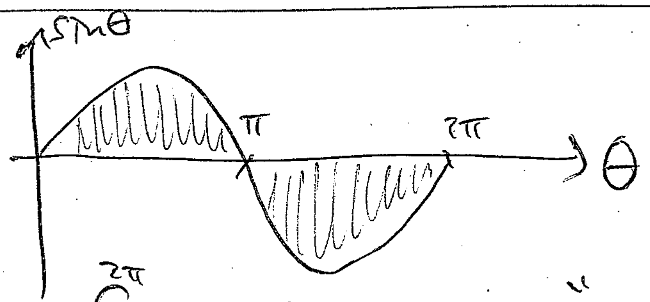
$$\Longrightarrow \iint_D xy\, dA = \int_{-2}^{4} \left(\int_{\frac{y^2-6}{2}}^{y+1} (x\, y)\, dx \right) dy$$

$$= \int_{-2}^{4} \left[\tfrac{1}{2} x^2 y \right]_{x=\frac{y^2-6}{2}}^{x=y+1} dy$$

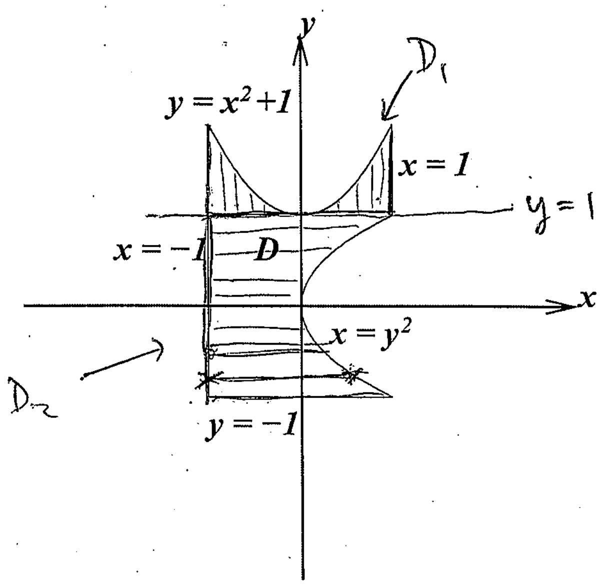
$$= \int_{-2}^{4} \left(\tfrac{1}{2}(y+1)^2 y - \tfrac{1}{2}\left(\frac{y^2-6}{2}\right)^2 y \right) dy$$

$$= 36.$$

= "net volume" above x–y plane. & below $z = f(x,y)$.



$$\int_0^{2\pi} \sin\theta\, d\theta = 0$$ "net area"



9.3 **Express** D **as a union of regions of type I or type II and expand the integral** $\iint_D f(x,y)\, dA$, **for some integrable function** f.

$D_1 = \{(x,y) \mid -1 \le x \le 1, \quad 1 \le y \le x^2+1\}$.

$D_2 = \{(x,y) \mid -1 \le y \le 1, \quad -1 \le x \le y^2\}$.

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

(properties of ~~integral~~ double int. p 51)

$$= \int_{-1}^{1}\int_{1}^{x^2+1} f \, dy\, dx + \int_{-1}^{1}\int_{-1}^{y^2} f \, dx\, dy$$