37 Complex matrices

By the end of this section, you should be able to answer the following questions:

- What are unitary, Hermitian and normal matrices?
- Given a complex matrix, determine if it can be unitarily diagonalised, and if so, diagonalise it.

Unitary and Hermitian matrices are complex analogues of orthogonal ($A^{-1} = A^T$) and symmetric ($A = A^T$) real matrices respectively.

In order to define these matrices, we need the following.

37.1 Definition (conjugate transpose)

$A^\dagger$

Let A be a complex matrix. The conjugate transpose of A, denoted A^*, is given by $(\overline{A})^T$, where $\overline{A}$ is the matrix whose entries are complex conjugates of the corresponding entries of A.

$z = a + ib$

Note that if A is real, $A^* = A^T$.

$\bar{z} = a - ib$.

37.1.1 Example

Let $A = \begin{pmatrix} 3+7i & 0 \\ 2i & 4-i \end{pmatrix}$. Write down the conjugate transpose of A.

$$\overline{A} = \begin{pmatrix} 3-7i & 0 \\ -2i & 4+i \end{pmatrix}$$

$$\Rightarrow A^* = (\overline{A})^T = \begin{pmatrix} 3-7i & -2i \\ 0 & 4+i \end{pmatrix}$$

37.2 Unitary matrices

A complex matrix A is said to be *unitary* if $A^{-1} = A^*$. Compare this definition with that of real orthogonal matrices. $(A^{-1} = A^T)$

Recall that a real matrix is orthogonal if and only if its columns form an orthonormal set of vectors. For complex matrices, this property characterises unitary matrices. In this case however, we must use the complex inner product.

37.3 Complex inner product

Recall that in $\mathbb{R}^n$ the inner (or dot) product of two vectors

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

is given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \quad = v^T u$$

and the length (a real number!) of $\boldsymbol{u}$ by

$$|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

These definitions are unsuitable for vectors in $\mathbb{C}^n$.

To demonstrate, consider the vector $\boldsymbol{u} = (i, 1)$ in $\mathbb{C}^2$. Using the above expression for length, we would obtain $|\boldsymbol{u}| = \sqrt{i^2 + 1} = 0$, so $\boldsymbol{u}$ would be a non-zero vector with length 0.

Instead, we introduce the complex inner product

$$\boxed{\boldsymbol{u} \cdot \boldsymbol{v} = u_1 \overline{v}_1 + u_2 \overline{v}_2 + \cdots + u_n \overline{v}_n,} \quad \in \mathbb{C} \text{ in general.}$$

where as usual $\overline{v}_i$ denotes the complex conjugate of v_i. In matrix notation, we can write this as $\boxed{\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v}^* \boldsymbol{u}}$. Note the length of a complex vector is always a real number.

So now we understand what is meant by the following statement: Columns of a unitary matrix form an orthonormal set with respect to the complex inner product.

- length $|\underset{\sim}{u}| = \sqrt{\underset{\sim}{u} \cdot \underset{\sim}{u}}$ (complex inner product)
- in case $\underset{\sim}{v}$ has only real entries, $\underset{\sim}{v}^* = \underset{\sim}{v}^T$

37.4 Hermitian (self-adjoint) matrices

A complex matrix A is called *Hermitian* (or *self-adjoint*) if $A = A^*$.

As with symmetric matrices, we can recognise a Hermitian matrix by inspection. See if you can see the pattern in the following 2×2, 3×3 and 4×4 Hermitian matrices.

$$\begin{pmatrix} a_{11} & a_{12} + ib_{12} \\ a_{12} - ib_{12} & a_{22} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & \boxed{a_{12} + ib_{12}} & a_{13} + ib_{13} \\ \boxed{a_{12} - ib_{12}} & a_{22} & a_{23} + ib_{23} \\ a_{13} - ib_{13} & a_{23} - ib_{23} & a_{33} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} + ib_{12} & a_{13} + ib_{13} & a_{14} + ib_{14} \\ a_{12} - ib_{12} & a_{22} & a_{23} + ib_{23} & a_{24} + ib_{24} \\ a_{13} - ib_{13} & a_{23} - ib_{23} & a_{33} & a_{34} + ib_{34} \\ a_{14} - ib_{14} & a_{24} - ib_{24} & a_{34} - ib_{34} & a_{44} \end{pmatrix},$$

where $a_{ij}, b_{ij} \in \mathbb{R}$. Note in particular that the diagonal entries are real numbers.

One of the most significant results on Hermitian matrices is that their eigenvalues are real.

Note: all real symmetric matrices are Hermitian.

37.4.1 Proof that Hermitian matrices have real eigenvalues

Let $v \in \mathbb{C}^n$ be an eigenvector of the Hermitian matrix A, with corresponding eigenvalue λ. In other words,

$$Av = \lambda v. \tag{13}$$

In what follows, we use the fact that $(AB)^* = B^*A^*$ which holds since the same is true for matrix transposition.

We multiply (13) from the left by v^* (treat v as an $n \times 1$ complex matrix) to obtain

$$v^*Av = v^*(\lambda v) = \lambda(v^*v). \tag{14}$$

Also note that

$$(v^*Av)^* = v^*A^*(v^*)^* = v^*Av.$$

In other words, v^*Av is also Hermitian. Since it evaluates to be a 1×1 matrix, and all Hermitian matrices have real numbers on their diagonal, this means that v^*Av is a real number.

The quantity v^*v is precisely the complex inner product of v with itself as we have already seen, which is also a real number.

Therefore equation (14) is of the form

$$x = \lambda y, \quad x, y \in \mathbb{R},$$

from which we must conclude that λ is real.

One consequence of this result is that a real symmetric matrix has real eigenvalues, since every real symetric matrix is Hermitian. This result was stated on page 212 but not proved.

37.5 Unitary diagonalisation

We have seen that real symmetric matrices are orthogonally diagonalisable. There is an analagous concept for complex matrices.

A square matrix A with complex entries is said to be unitarily diagonalisable if there is a unitary matrix P such that P^*AP is diagonal. , $AP = PD$.

It is natural to consider which matrices are unitarily diagonalisable. The answer lies in a more general class of matrix.

37.6 Normal matrices

A square complex matrix is called normal if it commutes with its own conjugate transpose, ie, if $\boxed{AA^* = A^*A.}$

Normal matrices are generally more difficult to identify by inspection. However, we have some classes of matrices which are normal:

- unitary, $AA^* = I = A^*A$.
- Hermitian, $A = A^* \Rightarrow AA^* = A^2 = A^*A$.
- real skew-symmetric (satisfying $A^T = -A$),
- any diagonal matrix,
- others?

We make a note that real normal 2×2 matrices are either symmetric or of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ (which include the skew-symmetric examples).

A class of matrix which is not generally normal is the class of complex symmetric matrices.

37.6.1 Example

Classify the matrix $A = \begin{pmatrix} 1 & 1+i \\ 1+i & -i \end{pmatrix}$.

$$A^* = \begin{pmatrix} 1 & 1-i \\ 1-i & i \end{pmatrix} \neq A \quad \text{(not Hermitian)}$$

$$A^*A = \begin{pmatrix} 1 & 1-i \\ 1-i & i \end{pmatrix} \begin{pmatrix} 1 & 1+i \\ 1+i & -i \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \neq I$$

($\Rightarrow$ not unitary)

but we have

$$A^*A = 3I \Rightarrow \tfrac{1}{3}A^* = A^{-1}.$$

$$\Rightarrow AA^* = 3I = A^*A.$$

$\Rightarrow A$ is <u>normal</u>.

37.7 Normal = unitarily diagonalisable

The main result we have is completely analagous to the real case of orthogonal diagonalisation and symmetric matrices on page 212. We will not prove this result.

An $n \times n$ complex matrix is unitarily diagonalisable if and only if it normal.

37.7.1 Example

If possible, diagonalise the matrix $\begin{pmatrix} 6 & 2+2i \\ 2-2i & 4 \end{pmatrix} = A$.

$A^* = \begin{pmatrix} 6 & 2+2i \\ 2-2i & 4 \end{pmatrix} = A \Rightarrow A$ is Hermitian.

$\Rightarrow A$ is normal $\Rightarrow A$ is unitarily diagonalisable.

$$\det\begin{pmatrix} 6-\lambda & 2+2i \\ 2-2i & 4-\lambda \end{pmatrix} = 24 - 10\lambda + \lambda^2 - 8$$

$$= \lambda^2 - 10\lambda + 16$$

$$= (\lambda - 8)(\lambda - 2)$$

$$\rightarrow \underline{\lambda = 8, 2}.$$

$\underline{\lambda = 8}$: $\begin{pmatrix} -2 & 2+2i \\ 2-2i & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

row 1. $\Rightarrow -2a + (2+2i)b = 0$

$\Rightarrow a = (1+i)b.$

$\underline{v}_1 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}.$

$\underline{\lambda = 2}$: $\underline{v}_2 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$

$\underset{\sim}{v}_1^* \underset{\sim}{v}_1 = (1-i \quad 1)\begin{pmatrix} 1+i \\ 1 \end{pmatrix} = 3.$

$\Rightarrow |\underset{\sim}{v}_1| = \sqrt{3}.$

& $|\underset{\sim}{v}_2| = \sqrt{3}.$

$\Rightarrow$ form $P = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & 1/\sqrt{3} \\ 1/\sqrt{3} & \frac{-1+i}{\sqrt{3}} \end{pmatrix}.$

check $PP^* = I = P^*P.$

then $P^*AP = D = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}.$