1. [**5 marks**] Calculate $\iint_D x\,dA$ where D is the region in the x-y plane bounded by $y = -2x + 3$ and $y = x^2$.

Ans. $= \dfrac{-32}{3}$

2. [**5 marks**] Evaluate the iterated integral (problem was changed)

$$\cancel{\int_0^1 \int_0^y \cos(x^2)\,dx\,dy} \quad \int_0^1 \int_y^1 \cos(x^2)\,dx\,dy$$

Ans. $= \frac{1}{2}(\sin 1)$

3. [**7 marks**] Use polar coordinates to evaluate the expression

$$\int_0^{1/2}\int_x^{\sqrt{3}x} xy\,dy\,dx + \int_{1/2}^{1/\sqrt{2}}\int_x^{\sqrt{1-x^2}} xy\,dy\,dx.$$

It may be helpful to sketch the regions of integration.

Ans. $= \dfrac{1}{32}$

4. [**7 marks**] Evaluate the iterated integral

$$\int_0^2 \int_0^y \int_0^{\sqrt{4-y^2}} 3x \, dx \, dz \, dy.$$

Ans. = 6

5. **[6 marks]** Let $f(x, y, z)$ be an integrable function. Find the functions $g(z)$ and $h(y, z)$ such that

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^{g(z)} \int_0^{h(y,z)} f(x,y,z)\,dx\,dy\,dz.$$

Ans.: $g(z) = 1 - z$

$h(y, z) = y^2$

6. **[10 marks]** Consider a planet of radius $r = a$ which is covered in a layer of ice of height $h = b - a$, so that the outer radius of the ice layer is $r = b$. The density of the ice is assumed to be $\rho(x, y, z) = \dfrac{kz^4}{(x^2 + y^2 + z^2)^2}$ for $a \le r \le b$. Find the mass of ice on the planet.

Ans.: $\dfrac{4\pi k}{15}(b^3 - a^3)$

7. **[15 marks]** Consider the velocity potential $\phi = \ln r = \ln\sqrt{x^2+y^2}$ in two dimensions.
   (a) Calculate the velocity field $\boldsymbol{v} = \nabla\phi$
   (b) Calculate $\nabla\cdot\boldsymbol{v}$
   (c) For a fluid of unit density, the rate at which mass flows out of a circle of radius a centered at the origin is given by $\oint \boldsymbol{v}\cdot\boldsymbol{n}ds$ where ds is an element of arclength. Evaluate this integral withiout using the divergence theorem.
   (d) Explain why the divergence theorem would fail in this case?

(a) $\underline{v} = \underline{\nabla}\phi = \dfrac{x}{x^2+y^2}\,\underline{i} + \dfrac{y}{x^2+y^2}\,\underline{j}$

(b) $\underline{\nabla}\cdot\underset{\sim}{v} = 0$

(c) 2π

(d) $\underline{v}$ not defined at $(x,y)=(0,0)$.

8. **[15 marks]** Consider the flux integral $\iint \mathbf{F}\cdot\mathbf{n}dS$ over the surface of sphere of radius a, where $\mathbf{F} = r^3\mathbf{r}$. On a sphere the unit normal vector is

$$\mathbf{n} = \hat{\mathbf{r}} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}.$$

(a) Evaluate the integral by using the Divergence theorem. You may assume $\mathrm{div}(r^n\mathbf{r}) = (n+3)r^n$.

(b) Evaluate the surface integral directly (do not use the divergence theorem).

(a) Ans.: $4\pi a^6$

(b) (same as (a))

9. **[5 marks]** Let $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$ $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$. Use the fact that $A = LL^T$ to solve $A\mathbf{x} = \mathbf{b}$.

Ans. : $\underline{x} = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

10. **[5 marks]** Let $A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 4 \\ 3 & -1 & -3 & 5 \\ 6 & -2 & -6 & 10 \end{bmatrix}$. Show that a r.e.f. of A is $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Write down a set of vectors which span the row space V_R of A, and a set of vectors which span the null space V_N of A. (The abbreviation r.e.f. stands for row echelon form).

Use Gaussian elimination to obtain r.e.f.

row space: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \\ 2 \end{pmatrix} \right\}$.

null space: $\left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

11. **[10 marks]** Let $C = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & 2 & 0 & -3 \\ 3 & -6 & -4 & 6 \\ -1 & 2 & 2 & -3 \end{bmatrix}$ $\quad \mathrm{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -6 \\ 3 \end{bmatrix}$.

(a) You are given that v_1 is an eigenvector of C. Find the associated eigenvalue.

(b) Find the eigenvectors associated with the eigenvalues $\lambda = -1$ and $\lambda = 2$.

(a) Ans. -7

(b) $\lambda = -1 \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 2 \longrightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

12. **[10 marks]** Let $C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad \mathrm{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\quad \mathrm{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \mathrm{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

The vectors v_i are the eigenvectors of C associated with the eigenvalues $\lambda_1 = 4$, $\lambda_2 = 3$, and $\lambda_3 = 1$ respectively.

(a) Write down a general solution to the system of differential equations

$$\begin{aligned} \dot{x}_1 &= 2x_1 + x_2 \\ \dot{x}_2 &= x_1 + 2x_2 \\ \dot{x}_3 &= 4x_3 \end{aligned}$$

where $\dot{x}_i = dx_i/dt$.

(b) Find a solution which satisfied the initial conditions $[x_1(0), x_2(0), x_3(0)]^T = [3, 1, 2]^T$.

(not examinable, although answer is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2e^{3t} + e^{t} \\ 2e^{3t} - e^{t} \\ e^{4t} \end{pmatrix}$$)