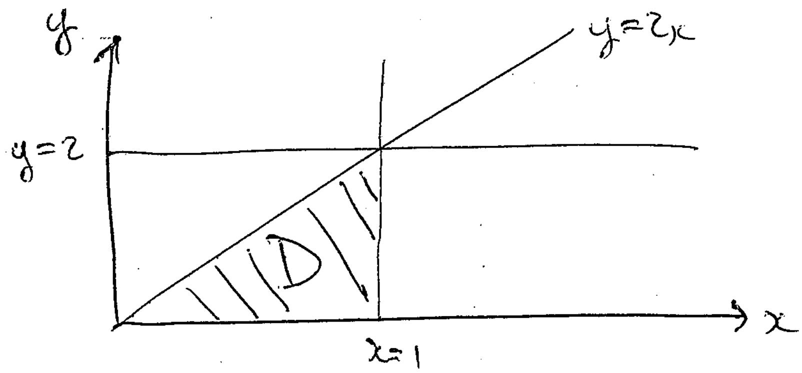
1. (a) Evaluate the integral $\int_1^2 \int_1^{x+1} 2y \, dy \, dx$.

Ans. $= \frac{16}{3}$

(b) Consider the integral $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy$. Plot the region of integration, change the order of integration and evaluate the integral.



$$\text{integral} = \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx$$

$$= e - 1.$$

2. Use cylindrical coordinates to find the mass of the region that lies above ($z \geq 0$) the circle $x^2 + y^2 = 1$ and below the sphere $x^2 + y^2 + z^2 = 4$ if the density is $\rho(x, y, z) = 2z$. Hint: Describe the sphere in cylindrical coodinates.

Ans. $= \frac{7\pi}{2}$

3. Consider a spherical planet of radius $r = a$ which has density $\rho(x, y, z) = k(x^2 + y^2)$ (the planet is centred at $(x, y, z) = (0, 0, 0)$). Find the mass of the planet. The identity $\sin^3 \phi = \sin \phi(1 - \cos^2 \phi)$ may be used.

Ans. $= \dfrac{8\pi k a^5}{15}$

4. Evaluate the line integral

$$\int_C (2xy+1)\ dx + (x^2+3y^2)\ dy,$$

where C is the curve from $(0,0)$ to $(0,2)$ consisting of the path from $(0,0)$ to $(1,1)$ along the parabola $y=x^2$, followed by the path along the straight line from $(1,1)$ to $(0,2)$.

Ans. = 8

5. Use Green's theorem to evaluate the line integral

$$\oint_C ((y^2 - x)dx + (3x + y)dy)$$

where C is the boundary of the region in the x-y plane enclosed by the parabola $y = x^2$ and the line $y = 1$, traversed in an anticlockwise direction.

What would the answer be if the curve was traversed in a clockwise direction?

Ans. $= \frac{12}{5}$, or $\frac{-12}{5}$ if traversed in clockwise direction.

6. Let the surface S be the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 2$. A parametrisation for S is given by the vector

$$\boldsymbol{r}(u, v) = u \cos v \boldsymbol{i} + u \sin v \boldsymbol{j} + u^2 \boldsymbol{k}, \ 0 \leq u \leq \sqrt{2}, \ 0 \leq v \leq 2\pi.$$

Find the surface area of S.

Ans. $= \dfrac{13\pi}{3}$

7. Consider the vector field $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + 2z\boldsymbol{k}$. Use the divergence theorem to find the net outward flux of $\boldsymbol{F}$ across the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Ans. $= \frac{16\pi}{3}$

8. Evaluate the line integral

$$\int_C y\ dx + x\ dy + 1\ dz,$$

along the path parametrised by $\boldsymbol{r}(t) = e^t\boldsymbol{i} + t^4\boldsymbol{j} + t\boldsymbol{k}$ for $0 \le t \le 1$.

Ans. $= e+1$

9. Let $\boldsymbol{F}(x,y,z) = (2y+1)\boldsymbol{i} - x\boldsymbol{k}$ and $\boldsymbol{G}(x,y,z) = xy\boldsymbol{i} - z\boldsymbol{j} + y^2\boldsymbol{k}$. Show that $\boldsymbol{F} = \operatorname{curl}\boldsymbol{G}$. Hence (or otherwise) use Stokes' theorem to evaluate the flux integral

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS,$$

where S is part of the open surface $z = 1 - x^2 - y^2$ which lies above the region in the x-y plane bounded by the unit circle centred at the origin, and S has upward orientation.

$$\text{Curl}\ \underline{G} = \underline{\nabla}\times\underline{G} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -z & y^2 \end{vmatrix}$$

$$= \underline{i}\,(2y+1) - \underline{j}\,(0) + \underline{k}\,(x)$$

$$= \underline{F}$$

Ans. : $\iint_S \underline{F}\cdot\underline{n}\,dS = 0$

10. Let $A = \begin{pmatrix} 4 & 3 & 2 \\ 12 & 12 & 8 \\ 8 & 12 & \alpha \end{pmatrix}$.

(a) Find an LU decomposition for the matrix for $\alpha \in \mathbb{R}$.

(b) For what value of α is the matrix A singular?

(a) $L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$

$U = \begin{pmatrix} 4 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & \alpha-8 \end{pmatrix}$

(b) $\alpha = 8$.

11. For the matrix $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 10 \end{pmatrix}$, use two steps of the power method, starting with the initial vector $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to give an estimate for the dominant eigenvalue of B and an approximation to the dominant eigenvector.

Ans. : dominant eigenvalue ≈ 10
dominant eigenvector $\approx \begin{pmatrix} 0.09 \\ 0.09 \\ 1 \end{pmatrix}$

(NOT EXAMINABLE)

12. (a) Find an orthogonal matrix which diagonalises the matrix $\begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$. The characteristic polynomial of this matrix is $(\lambda - 9)(\lambda - 4) = 0$.

(b) The equation $5x^2 - 4xy + 8y^2 = 36$ can be expressed as $\frac{u^2}{9} + \frac{v^2}{4} = 1$ (the equation of an ellipse) where u and v are functions of x and y. Use your answer to part (a) to give the functions $u(x, y)$ and $v(x, y)$.

(a) Ans: orthogonal matrix

$$P = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

(b) Ans:

$$u(x,y) = \frac{2x+y}{\sqrt{5}}$$

$$V(x,y) = \frac{x-2y}{\sqrt{5}}$$