DEPARTMENT OF MATHEMATICS

MATH2000
Revision questions

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1. Evaluate the integral $\int_0^1 \int_{y^2}^1 \exp x^{\frac{3}{2}} dx dy$.

2. Find the mass of the portion of the sphere of radius a centred at the origin which lies in the positive octant ($x \geq 0$, $y \geq 0$ and $z \geq 0$). The density of the sphere in this region is given by $\rho(x, y, z) = kxz$ kg/m^3 where k is a constant.

3. Use cylindrical coordinates to find the mass of the solid of density e^z which lies in the closed region $0 \leq z \leq 4 - x^2 - y^2$.

4. Evaluate the line integral
$$\int_C y\ dx + 3y^2\ dy,$$
where C is the straight line from $(1, 2)$ to $(0, 0)$.

5. Evaluate the line integral
$$\int_C (2x + 1)\ dx + 2y\ dy,$$
where C is the curve from $(0, 1)$ to $(\pi/2, 0)$ along the curve $y = \cos x$.

6. Evaluate the line integral
$$\int_C 2xy\ dx + x^2\ dy,$$
where C is the curve from $(1, 1)$ to $(2, 0)$ consisting of the path from $(1, 1)$ to $(0, 0)$ along the parabola $y = x^2$, followed by the path along the x-axis from $(0, 0)$ to $(2, 0)$.

7. Use Green's theorem to find the work done by the force
$$\boldsymbol{F}(x, y) = x(x + y)\boldsymbol{i} + xy^2\boldsymbol{j}$$
in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

8. Use Green's theorem to evaluate the line integral
$$\oint_C \left(y^3\ dx - x^3\ dy\right)$$
where C is the circle $x^2 + y^2 = 4$ traversed in an anticlockwise direction.

9. Find the net outward flux of $\boldsymbol{F}(x, y) = x\boldsymbol{i} + y\boldsymbol{j}$ across the circle parametrised by
$$\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j},\ 0 \leq t \leq 2\pi.$$

10. Find a parametric representation for the surface S, where S is part of the plane $z = x+3$ that lies inside the cylinder $x^2 + y^2 = 1$. Find the surface area of the surface S.

11. A triangular surface is given by the parametrisation

$$x(u, v) = 6(1 - v),\ y(u, v) = 3uv,\ z(u, v) = 2(1 - u)v,\ 0 \le u \le 1,\ 0 \le v \le 1.$$

Find the average value of the function $f(x, y, z) = x + y + z$ on the surface.

12. Use Gauss' divergence theorem to calculate the net outward flux of the vector field

$$\boldsymbol{F}(x, y, z) = x^4\boldsymbol{i} - x^3z^2\boldsymbol{j} + 4xy^2z\boldsymbol{k}$$

across the surface of the solid region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$.

13. Use Gauss' divergence theorem to evaluate $\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$, where

$$\boldsymbol{F}(x, y, z) = z^2x\boldsymbol{i} + (\frac{1}{3}y^3 + \tan z)\boldsymbol{j} + (x^2z + y^2)\boldsymbol{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ with upward orientation. (*Hint:* Note that S is not a closed surface. First compute integrals over S_1 and S_2 where S_1 is the disk $x^2 + y^2 \le 1$, oriented downward, and $S_2 = S \cup S_1$.)

14. Evaluate the line integral $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$ where

$$\boldsymbol{F}(x, y, z) = yz\boldsymbol{i} + xz\boldsymbol{j} + (xy + 2z)\boldsymbol{k}$$

and C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

15. Evaluate the line integral $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$ where

$$\boldsymbol{F}(x, y, z) = e^y\boldsymbol{i} + xe^y\boldsymbol{j} + (z + 1)e^z\boldsymbol{k}$$

and C is the curve parametrised by

$$\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k},\ 0 \le t \le 1.$$

16. Use Stokes' theorem to evaluate $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$, where

$$\boldsymbol{F}(x, y, z) = (x + y^2)\boldsymbol{i} + (y + z^2)\boldsymbol{j} + (z + x^2)\boldsymbol{k}$$

and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ oriented in a counterclockwise direction as viewed from above.

17. Use Stokes' theorem to evaluate $\iint_S (\text{curl}\,\boldsymbol{F}) \cdot \boldsymbol{n}\; dS$, where

$$\boldsymbol{F}(x, y, z) = xyz\boldsymbol{i} + xy\boldsymbol{j} + x^2yz\boldsymbol{k}$$

and S consist of the top and four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented upward.

18. Let $A = \begin{pmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix}$, $L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ -3 \\ 10 \end{pmatrix}$. Use the fact that $A = LL^T$ to solve $Ax = b$. What is the determinant of A?

19. Find an LU decomposition of the matrix $A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 4 & 0 & 2 & 1 \\ 8 & 9 & 6 & 3 \\ 4 & 6 & 4 & 2 \end{pmatrix}$ and hence write down its determinant.

20. Find a PLU decomposition of the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and hence write down its determinant.

21. Find the general solution to the O.D.E.

$$\cos y \sinh x + 1 - \sin y \cosh x \frac{dy}{dx} = 0.$$

22. Find the general solution to the initial value problem

$$e^{-2\theta} \frac{dr}{d\theta} - 2re^{-2\theta} = 0, \quad r(0) = 1.$$

23. The equation of a conic section is given by $3x^2 - 2xy + 3y^2 - 2x - 2y - 4 = 0$. Translate and rotate the coordinate axes, if necessary, to put the conic section into standard form. Name the conic and give its equation in the final coordinate system.

24. Solve the inital value problem $y'' + 2y' + y = 2\cosh x$, $y(0) = 3/4$, $y'(0) = 1/4$.

25. Solve the initial value problem $6\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + y = t - 10\sin t$, $y(0) = 9$, $\dfrac{dy}{dt}(0) = 4$.

26. Solve the initial value problem $y'' + 4y' + 4y = 6te^{-2t}$, $y(0) = -1$, $y'(0) = 2$.

27. You are given that $y = x^2$ is a solution to the differential equation $x^2y'' + 2xy' - 6y = 0$. Find the general solution.

28. Use a hyperbolic function as a substitution to evaluate the integral

$$\int \frac{3x\ dx}{\sqrt{x^4 - 9}}.$$

29. Use a hyperbolic function as a substitution to evaluate the integral

$$\int \frac{4\ dx}{\sqrt{x^2 + 16}}.$$

30. Is the vector field represented in the following diagram conservative? Explain your answer.