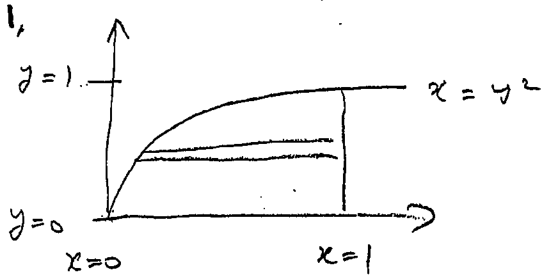
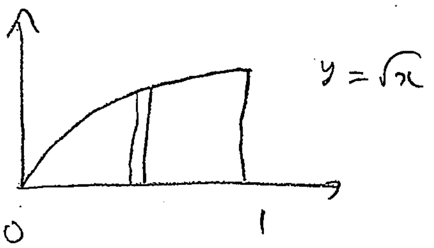
# MATH2000 Revision Solutions.

1.




$(y = \sqrt{x})$




$$\int_0^1 \int_0^{\sqrt{x}} e^{x^{3/2}} \, dy \, dx$$

$$= \int_0^1 e^{x^{3/2}} \, y \Big|_0^{\sqrt{x}} \, dx$$

$$= \int_0^1 e^{x^{3/2}} x^{1/2} \, dx$$

$u = x^{3/2} \qquad du = \frac{3}{2} x^{1/2} \, dx$

$\frac{2}{3} du = x^{1/2} dx$

$$= \int e^{u} \tfrac{2}{3} \, du = \tfrac{2}{3} e^{u} + c$$

$$\tfrac{2}{3} e^{x^{3/2}} \Big|_0^1 = \tfrac{2}{3}(e - 1)$$

2. $$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a k\underbrace{r\cos\theta\sin\phi}_{x}\,\underbrace{r\cos\phi}_{z}\,\underbrace{r^2\sin\phi\,dr\,d\theta\,d\phi}_{dr}$$

$$= k\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi\int_0^{\pi/2}\cos\theta\,d\theta\int_0^a r^4\,dr$$

$u = \sin\phi$

$du = \cos\phi\,d\phi$

$$\int \sin^2\phi\cos\phi\,d\phi = \int u^2\,du = \frac{1}{3}u^3 + c$$

$$= k\ \frac{1}{3}\sin^3\phi\Big|_0^{\pi/2}\cdot\sin\theta\Big|_0^{\pi/2}\ \frac{1}{5}a^5$$

$$= k\cdot\frac{1}{3}\cdot\left(\sin\tfrac{\pi}{2}-\sin 0\right)\cdot\frac{1}{5}a^5$$

$$= \frac{1}{15}ka^5$$

3. $$\int_0^{2\pi}\int_0^2\int_0^{z=4-x^2-y^2} e^z \, r\,dz\,dr\,d\theta$$

on $x$-$y$ plane
$0 = 4-x^2-y^2$
$x^2+y^2=4$
or $r=2$

Express upper boundary in terms of $r$ & $\theta$

$$z = 4 - r^2\cos^2\theta - r^2\sin^2\theta = 4-r^2$$

$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} e^z \, dz\, r\,dr\, d\theta$$

$$\int_0^{2\pi}\int_0^2 e^z\Big|_0^{4-r^2} r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^2 \left(e^{4-r^2}-e^0\right) r\,dr\,d\theta$$

$$\int_0^{2\pi} d\theta \quad -\frac{e^{4-r^2}}{2} - \frac{1}{2}r^2\Big|_0^2$$

$$2\pi\left[\left(-\frac{1}{2}-2\right)-\left(-\frac{e^4}{2}-0\right)\right]$$

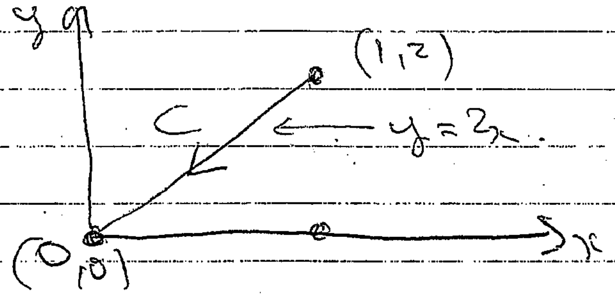
$$= 2\pi\left(\frac{e^4}{2}-\frac{5}{2}\right) = \pi\left(e^4-5\right)$$

4. $\int_C y\,dx + 3y^2\,dy$.

$F_1 = y$, $F_2 = 3y^2$

Is $\underline{F} = F_1\underline{i} + F_2\underline{j}$ conservative?

$\frac{\partial F_1}{\partial y} = 1$, $\frac{\partial F_2}{\partial x} = 0$ $\Rightarrow$ not conservative.




parametrise C: $\underline{r}(t) = (1-t)\underline{i} + (2-2t)\underline{j}$

$0 \le t \le 1$.

$x = 1-t \Rightarrow dx = -dt$.

$y = 2-2t \Rightarrow dy = -2dt$.

$$\Rightarrow \int_C y\,dx + 3y^2\,dy = \int_0^1 (2-2t)(-dt) + 3(2-2t)^2(-2dt)$$

$$= \int_0^1 \left(2t - 2 - 6(4-8t+4t^2)\right) dt$$

$$= \int_0^1 (-24t^2 + 50t - 26)\,dt$$

$$= -8 + 25 - 26$$

$$= -9.$$

5. $F_1 = 2x+1$ , $F_2 = 2y$ .

$$\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x} \Rightarrow \underline{F} \text{ is conservative}$$

$$\Rightarrow \exists f \text{ s.t. } \underline{F} = \underline{\nabla} f$$

i.e. $\frac{\partial f}{\partial x} = 2x+1$ — (1)

$\frac{\partial f}{\partial y} = 2y$ — (2)

(1) $\Rightarrow f(x,y) = x^2 + x + g(y)$

$\Rightarrow \frac{\partial f}{\partial y} = g' = 2y$ from (2) .

$\Rightarrow g(y) = y^2 + c$ .

$\Rightarrow f(x,y) = x^2 + x + y^2 + c$

By the fundamental theorem for line integrals

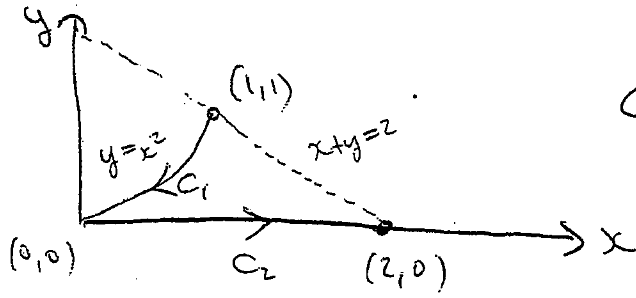
$$\int_C \underline{\nabla} f \cdot d\underline{r} = f(\text{end point}) - f(\text{start point})$$

$$= f\left(\tfrac{\pi}{2}, 0\right) - f(0,1)$$

$$= \frac{\pi^2}{4} + \frac{\pi}{2} + c - (1+c)$$

$$\Rightarrow \int_C (2x+1)dx + 2y\,dy = \frac{\pi^2}{4} + \frac{\pi}{2} - 1 .$$

6.




$C = C_1 \cup C_2$

$$\left.\begin{aligned} F_1 &= 2xy \Rightarrow \frac{\partial F_1}{\partial y} = 2x \\ F_2 &= x^2 \Rightarrow \frac{\partial F_2}{\partial x} = 2x \end{aligned}\right\} \Rightarrow \underline{F} = F_1 \underline{i} + F_2 \underline{j} \text{ is conservative.}$$

$\Rightarrow$ line integrals are path independent.
For simplicity, take the straight line from (1,1) to (2,0), ie. along $x+y=2$

Parametrise curve:

$$\underline{r}(t) = t\underline{i} + (2-t)\underline{j} \qquad 1 \le t \le 2$$

$$\Rightarrow \int_C 2xy\,dx + x^2\,dy$$

$$= \int_1^2 2(2t-t^2)\,dt + \int_1^2 t^2(-dt)$$

$$= 2\left[t^2 - \tfrac{1}{3}t^3\right]_1^2 - \left[\tfrac{1}{3}t^3\right]_1^2$$

$$= \left[2t^2 - t^3\right]_1^2$$

$$= 8 - 8 - (2-1) = -1.$$

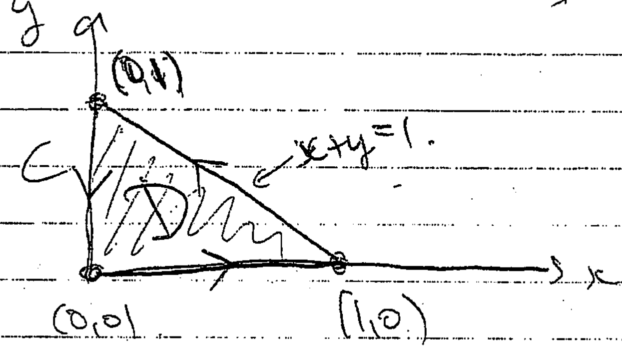
(OR... could have determined the potential function $f(x,y) = x^2y + c$ )

Q7. work done by $\underline{F} = \oint_C \underline{F} \cdot d\underline{r}$.

$$\underline{F} = x(x+y)\,\underline{i} + xy^2\,\underline{j}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y^2 - x.$$

By Greens theorem $\oint_C \underline{F} \cdot d\underline{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$




$$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}.$$

$$\Rightarrow \oint_C \underline{F} \cdot d\underline{r} = \int_0^1 \int_0^{1-x} (y^2 - x)\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{3}y^3 - xy\right]_0^{1-x} dx.$$

$$= \int_0^1 \left(\frac{1}{3}(1-x)^3 - x(1-x)\right) dx.$$

$$\boxed{\begin{array}{l}\text{set } u = 1-x \Rightarrow du = -dx. \\ (x=0,\ u=1)\ \&\ (x=1,\ u=0)\end{array}}$$

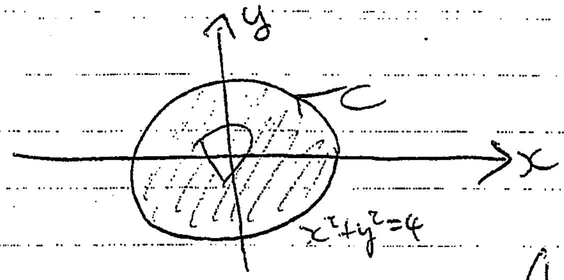
$$= \int_{u=1}^{u=0} \left(\frac{1}{3}u^3 - (1-u)u\right)(-du)$$

$$= \int_0^1 \left(\frac{1}{3}u^3 + u^2 - u\right) du = \frac{1}{12} + \frac{1}{3} - \frac{1}{2}$$

$$= \frac{1+4-6}{12} = \frac{-1}{12}.$$

(= work done).

8.




$$\oint_C y^3\,dx - x^3\,dy$$

$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

(by Green's theorem)

$\frac{\partial F_1}{\partial y} = 3y^2$, $\frac{\partial F_2}{\partial x} = -3x^2$

In polar coords, $D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$.

& $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2+y^2 = r^2$

$$\Rightarrow \oint_C y^3\,dx - x^3\,dy = \iint -3(x^2+y^2)\,dA$$

$$= -3\int_0^2 \int_0^{2\pi} r^2 \cdot r\,d\theta\,dr$$

$$= -3\left[\frac{1}{4}r^4\right]_0^2 \times 2\pi$$

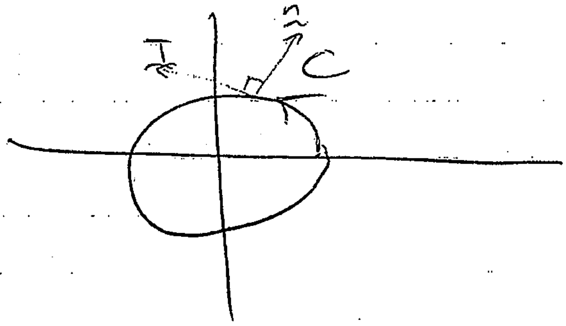
$$= -24\pi$$

9. $C: \underline{r}(t) = \cos t\,\underline{i} + \sin t\,\underline{j}$

$\Rightarrow \underline{r}'(t) = -\sin t\,\underline{i} + \cos t\,\underline{j}$

which is a unit tangent vector, so take

$$\underline{T}(t) = \frac{\underline{r}'(t)}{|\underline{r}'(t)|} = -\sin t\,\underline{i} + \cos t\,\underline{j}$$




Outwardly pointing unit normal vector $\underline{n}$ : (use right hand rule)

$$\underline{n}(t) = \underline{T} \times \underline{k} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \underline{i}(\cos t) - \underline{j}(-\sin t) + \underline{k}(0)$$

$$= \cos t\,\underline{i} + \sin t\,\underline{j}$$

We now calculate $\oint_C \underline{F}\cdot\underline{n}\,ds$ :

$\underline{F}(\underline{r}(t)) = \cos t\,\underline{i} + \sin t\,\underline{j}$

also, in $\int$, $ds = |\underline{r}'(t)|\,dt \Rightarrow ds = dt$

$\Rightarrow \underline{F}(\underline{r}(t))\cdot\underline{n}(t) = \cos^2 t + \sin^2 t = 1$

$$\Rightarrow \oint_C \underline{F}\cdot\underline{n}\,ds = \int_0^{2\pi} \underline{F}(\underline{r}(t))\cdot\underline{n}(t)\,|\underline{r}'(t)|\,dt$$

$$= \int_0^{2\pi} dt = 2\pi.$$

Q. 10. Part of the plane $z = x+3$ inside $x^2+y^2=1$

i.e. $x^2+y^2 \leq 1$

$\Rightarrow$ Use parametrisation based on cylindrical (or polar) coordinates

$$S: \quad x = r\cos\theta \qquad 0 \leq r \leq 1$$
$$y = r\sin\theta \qquad 0 \leq \theta \leq 2\pi$$
$$z = 3 + r\cos\theta$$

$$\Rightarrow \underline{r}(r,\theta) = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + (3+r\cos\theta)\,\underline{k}$$

$$\underline{r}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} + \cos\theta\,\underline{k}$$
$$\underline{r}_\theta = -r\sin\theta\,\underline{i} + r\cos\theta\,\underline{j} - r\sin\theta\,\underline{k}$$

$$\Rightarrow \underline{r}_r \times \underline{r}_\theta = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & \cos\theta \\ -r\sin\theta & r\cos\theta & -r\sin\theta \end{vmatrix}$$
$$= \underline{i}\,(-r\sin^2\theta - r\cos^2\theta) - \underline{j}\,(-r\sin\theta\cos\theta + r\sin\theta\cos\theta) + \underline{k}\,(r\cos^2\theta + r\sin^2\theta)$$
$$= -r\,\underline{i} + r\,\underline{k}$$

$$|\underline{r}_r \times \underline{r}_\theta| = \sqrt{r^2+r^2} = r\sqrt{2}$$

Here we use surface area $= \iint_S dS$

$$= \iint_D |\underline{r}_r \times \underline{r}_\theta|\, dr\, d\theta$$
$$= \int_0^1 \int_0^{2\pi} r\sqrt{2}\, d\theta\, dr$$
$$= \sqrt{2} \times 2\pi \times \left[\tfrac{1}{2} r^2\right]_0^1$$
$$= \pi\sqrt{2}.$$

11. $S: \underline{r}(u,v) = 6(1-v)\,\underline{i} + 3uv\,\underline{j} + 2(1-u)v\,\underline{k}$

$0 \le u \le 1, \quad 0 \le v \le 1$

$\underline{r}_u = 3v\,\underline{j} - 2v\,\underline{k}$

$\underline{r}_v = -6\,\underline{i} + 3u\,\underline{j} + 2(1-u)\,\underline{k}$

$$\Rightarrow \underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 3v & -2v \\ -6 & 3u & 2(1-u) \end{vmatrix}$$

$$= \underline{i}\,(6(1-u)v + 6uv) - \underline{j}\,(-12v) + \underline{k}\,(18v)$$

$$= 6v\,\underline{i} + 12v\,\underline{j} + 18v\,\underline{k}$$

$$\Rightarrow |\underline{r}_u \times \underline{r}_v| = \sqrt{(6^2+12^2+18^2)\,v^2} = \sqrt{6^2(1+2^2+3^2)v^2} = 6\sqrt{14}\,v.$$

$$\text{surface area} = \iint_S dS = 6\sqrt{14}\int_0^1\int_0^1 v\,du\,dv$$

$$= 6\sqrt{14} \times 1 \times \tfrac{1}{2} = 3\sqrt{14}.$$

$$\text{average value of } f(x,y,z) = x+y+z \text{ over } S = \frac{\iint_S f(x,y,z)\,dS}{(\text{surface area of } S)}$$

$$\iint_S f(x,y,z)\,dS = \int_0^1\int_0^1 f(x(u,v), y(u,v), z(u,v))\,|\underline{r}_u \times \underline{r}_v|\,du\,dv$$

$$= 6\sqrt{14}\int_0^1\int_0^1 (\underbrace{6-6v}_{x} + \underbrace{3uv}_{y} + \underbrace{2v-2uv}_{z})\,v\,du\,dv$$

$$= 6\sqrt{14}\int_0^1\int_0^1 (6v - 4v^2 + uv^2)\,du\,dv$$

$$= 6\sqrt{14}\int_0^1 \left[6uv - 4uv^2 + \tfrac{1}{2}u^2v^2\right]_0^1 dv$$

$$= 6\sqrt{14}\int_0^1 (6v - 4v^2 + \tfrac{1}{2}v^2)\,dv$$

$$= 6\sqrt{14}\left[3v^2 - \tfrac{4}{3}v^3 + \tfrac{1}{6}v^3\right]_0^1$$

$$= 6\sqrt{14}\left(3 - \tfrac{4}{3} + \tfrac{1}{6}\right)$$

$$= 6\sqrt{14}\left(\frac{18-8+1}{6}\right) = 11\sqrt{14}$$

$$\Rightarrow \text{average value} = \frac{11\sqrt{14}}{3\sqrt{14}} = \frac{11}{3}.$$

Q.12. Net outward flux of $\underline{F}$ across closed surface $S$ $= \iint_S \underline{F}\cdot\underline{n}\, dS$ ($\underline{n}$ - outwardly directed unit normal vector)

$= \iiint_V \operatorname{div} \underline{F}\, dV$ by Gauss' divergence theorem, since $\underline{F}$ is well defined and continuous everywhere on and within S.

$\underline{F} = x^4 \underline{i} - x^3 z^2 \underline{j} + 4xy^2 z \underline{k}$

$\Rightarrow \operatorname{div} \underline{F} = 4x^3 + 4xy^2 = 4x(x^2+y^2)$

For V (volume bounded by S: cylinder $x^2+y^2=1$, $z=x+2$, $z=0$), use cylindrical coords $x = r\cos\theta$, $y = r\sin\theta$, $z$.

$\Rightarrow \operatorname{div} \underline{F} = 4r\cos\theta \cdot r^2 = 4r^3\cos\theta$

$0 \le r \le 1$, $0 \le \theta \le 2\pi$, $0 \le z \le x+2$

$\Rightarrow 0 \le z \le 2 + r\cos\theta$.

$$\Rightarrow \iiint_V \operatorname{div} \underline{F}\, dV = \int_0^1 \int_0^{2\pi} \int_0^{2+r\cos\theta} 4r^3\cos\theta \cdot r\, dz\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} 4r^4\cos\theta\,(2 + r\cos\theta)\, d\theta\, dr$$

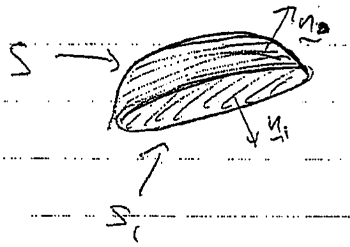
(use $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$)

$$= \int_0^1 \int_0^{2\pi} \Big(\underbrace{\cancel{8r^4\cos\theta}}_{=0} + 4r^5\big(\tfrac{1}{2} + \underbrace{\cancel{\tfrac{1}{2}\cos 2\theta}}_{=0}\big)\Big)\, d\theta\, dr$$

$$= 2\int_0^1 \int_0^{2\pi} r^5\, d\theta\, dr$$

$$= 2 \times \frac{1}{6} \times 2\pi = \frac{2}{3}\pi.$$

13. Let $S_1$ be disk $x^2+y^2\leq 1$ in the $xy$ plane.
& $S_2$ be the closed surface




$S_2 = S \cup S_1$

$$\iint_{S_2} \underline{F}\cdot \underline{n_2}\,dS = \iiint_V \operatorname{div} \underline{F}\, dV$$

$V$ : solid hemisphere with surface $S_2$

(use spherical coords) $= \{(r,\phi,\theta) \mid 0\leq r\leq 1,\ 0\leq \phi\leq \frac{\pi}{2},\ 0\leq\theta\leq 2\pi\}$.

$$\underline{F} = z^2x\,\underline{i} + \left(\tfrac{1}{3}y^3+\tan z\right)\underline{j} + (x^2z+y^2)\,\underline{k}$$

$$\Rightarrow \operatorname{div}\underline{F} = z^2+y^2+x^2.$$

$$= r^2 \quad \text{(in spherical coords)}$$

$$\Rightarrow \iiint_V \operatorname{div}\underline{F}\,dV = \int_0^1\int_0^{\pi/2}\int_0^{2\pi} r^2\cdot r^2\sin\phi\, d\theta\, d\phi\, dr.$$

$$= \left(\int_0^1 r^4 dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2}\sin\phi\, d\phi\right)$$

$$= \frac{1}{5}\times 2\pi \times \underbrace{\left[-\cos\phi\right]_0^{\pi/2}}_{=1}$$

$$= \frac{2\pi}{5}.$$

$\iint_{S_1} \underline{F}\cdot\underline{n_1}\,dS$ in this case $\underline{n}_1 = -\underline{k}$.

$S_1 = \{z=0,\ x^2+y^2\leq 1\}$,

$$\Rightarrow \underline{F} = \tfrac{1}{3}y^3\,\underline{j} + y^2\,\underline{k}.$$

$$\Rightarrow \underline{F}\cdot(-\underline{k}) = -y^2$$

use polar coords : $S_1 = \{(r,\theta) \mid 0\leq r\leq 1,\ 0\leq\theta\leq 2\pi\}$

$$\underline{F}\cdot(-\underline{k}) = -y^2 = -r^2\sin^2\theta.$$

$$\Rightarrow \iint_{S_1}\underline{F}\cdot\underline{n}_1\,dS = -\int_0^1\int_0^{2\pi} r^2\sin^2\theta\cdot r\,d\theta\,dr$$

$$= -\left(\int_0^1 r^3 dr\right)\left(\int_0^{2\pi}\sin^2\theta\, d\theta\right)$$

$$= -\frac{1}{4}\cdot\int_0^{2\pi}\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$= -\pi/4. \Rightarrow \iint_S \underline{F}\cdot\underline{n} = \iint_{S_2}\underline{F}\cdot\underline{n_2}\,dS - \iint_{S_1}\underline{F}\cdot\underline{n}_1\,dS = \frac{2\pi}{5}+\frac{\pi}{4}$$

$$= 13\pi/20.$$

14. Check if $\underline{F}$ is conservative.

$$\operatorname{curl} \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy+2z \end{vmatrix}$$

$$= \underline{i}(x-x) - \underline{j}(y-y) + \underline{k}(z-z) = \underline{0}$$

$\Rightarrow \underline{F}$ is conservative.

$\Rightarrow \exists f$ s.t. $\underline{F} = \underline{\nabla} f$

ie $\dfrac{\partial f}{\partial x} = yz$ — (1)

$\dfrac{\partial f}{\partial y} = xz$

$\dfrac{\partial f}{\partial z} = xy + 2z$

(1) $\Rightarrow f = xyz + g(y,z)$

$\Rightarrow \dfrac{\partial f}{\partial y} = xz + \dfrac{\partial g}{\partial y} = xz$ from (2)

$\Rightarrow \dfrac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

$\Rightarrow f = xyz + h(z)$

$\dfrac{\partial f}{\partial z} = xy + h'(z) = xy + 2z$

$\Rightarrow h'(z) = 2z \Rightarrow h(z) = z^2 + c$.

$\Rightarrow f(x,y,z) = xyz + z^2 + c$.

(fundamental theorem for line integrals) $\Rightarrow \displaystyle\int_C \underline{\nabla} f \cdot d\underline{r} = f(\text{end point}) - f(\text{start point})$

$= f(4,6,3) - f(1,0,-2)$

$= 72 + 9 - 4$

$= 77$.

15. check if $\underline{F}$ is conservative

$$\text{curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & xe^y & (z+1)e^z \end{vmatrix}$$

$$= \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(e^y - e^y)$$

$$= \underline{0}$$

$\Rightarrow$ $\underline{F}$ is conservative

$\Rightarrow \exists f$ s.t. $\underline{F} = \nabla f$

ie $\frac{\partial f}{\partial x} = e^y$ — (1)

$\frac{\partial f}{\partial y} = xe^y$ — (2)

$\frac{\partial f}{\partial z} = (z+1)e^z$ — (3).

C is from $(0,0,0)$ to $(1,1,1)$ (ie $\underline{r}(0)$ to $\underline{r}(1)$) where $\underline{r}(t) = t\underline{i} + t^2\underline{j} + t^3\underline{k}$.

(1) $\Rightarrow f = xe^y + g(y,z)$

$\Rightarrow \frac{\partial f}{\partial y} = xe^y + \frac{\partial g}{\partial y}$

$= xe^y$ from (2).

$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

$\Rightarrow f = xe^y + h(z)$

$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = (z+1)e^z$ from (3).

$\Rightarrow h(z) = \int \overset{u}{(z+1)}\overset{v'}{e^z}\,dz$. $\quad \int uv' = uv - \int u'v$

$= (z+1)e^z - \int e^z dz$

$= ze^z + c$

$\Rightarrow f(x,y,z) = xe^y + ze^z + c$.

By fundamental theorem of line integrals,

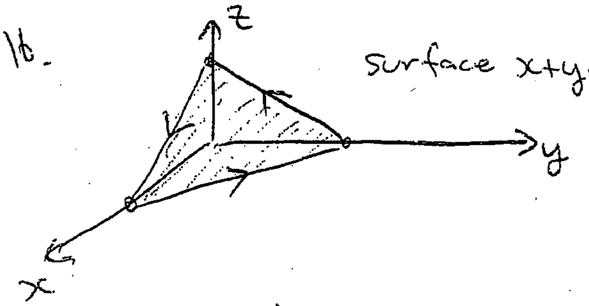
$\int_C \nabla f \cdot d\underline{r} = f(\text{end point}) - f(\text{start point})$

$= f(1,1,1) - f(0,0,0)$

$= e + e + c - (0 + c)$

$= 2e$.

Q. 16.




surface $x+y+z=1$

$$\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\text{curl}\,\underline{F})\cdot\underline{n}\,dS$$

by Stokes' theorem.

$$\text{curl}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$$

$$= \underline{i}(0-2z) - \underline{j}(2x) - \underline{k}(2y)$$

Parametrise $S$: $\underline{r}(x,y) = x\underline{i} + y\underline{j} + (1-x-y)\underline{k}$

over $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$\underline{r}_x = \underline{i} - \underline{k}$

$\underline{r}_y = \underline{j} - \underline{k}$

$$\underline{r}_x \times \underline{r}_y = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \underline{i} + \underline{j} + \underline{k}$$

(note direction is okay)

$$\text{curl}\,\underline{F} \text{ on } S = -2(z\underline{i} + x\underline{j} + y\underline{k})$$
$$= -2((1-x-y)\underline{i} + x\underline{j} + y\underline{k})$$

$$(\text{curl}\,\underline{F})\cdot(\underline{r}_x \times \underline{r}_y) = -2(1-x-y+x+y) = -2$$

$$\Rightarrow \iint_S (\text{curl}(\underline{F}))\cdot\underline{n}\,dS = \iint_D (\text{curl}\,\underline{F})\cdot(\underline{r}_x\times\underline{r}_y)\,dx\,dy$$

$$= -2\int_0^1\int_0^{1-x} dy\,dx$$

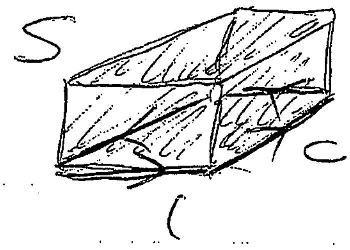
$$= -2\int_0^1 (1-x)\,dx$$

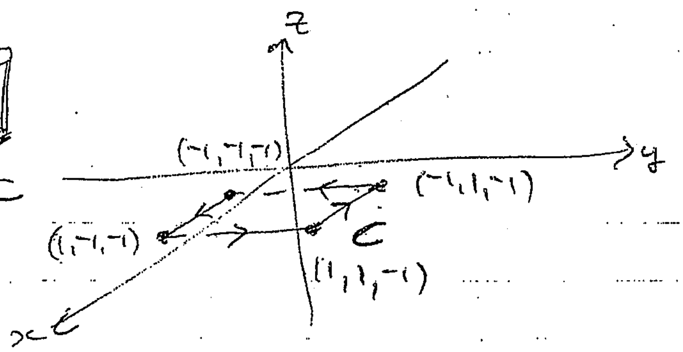
$$= -2\left[x - \tfrac{1}{2}x^2\right]_0^1 = -2\cdot\tfrac{1}{2} = -1.$$

(Note we have used variables $x, y$ to parametrise the surface in this case).

~~#~~ 17. Stokes' theorem: $\iint_S (\text{curl}\,\underline{F})\cdot\underline{n}\,dS = \oint_C \underline{F}\cdot d\underline{r}$

where C is the bounding curve of the open surface S.




Let S' be the surface forming the base of the box. That is, in the plane $z=-1$, $\{-1\le x\le 1, -1\le y\le 1\}$.
By Stokes' theorem,

$$\iint_{S'} (\text{curl}\,\underline{F})\cdot\underline{n}\,dS = \oint_C \underline{F}\cdot d\underline{r}$$
$$= \iint_S (\text{curl}\,\underline{F})\cdot\underline{n}\,dS.$$

In this case $\underline{F} = xyz\,\underline{i} + xy\,\underline{j} + x^2yz\,\underline{k}$.

$$\Rightarrow \text{curl}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy & x^2yz \end{vmatrix}$$
$$= \underline{i}\,(x^2z - 0) - \underline{j}\,(2xyz - xy) + \underline{k}\,(y - xz)$$
$$= x^2z\,\underline{i} - xy(2z-1)\,\underline{j} + (y - xz)\,\underline{k}.$$

restricted to the plane $z=-1$

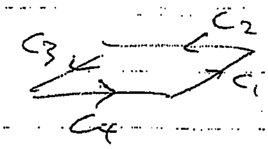
$$\Rightarrow \text{curl}\,\underline{F}\Big|_{z=-1} = -x^2\,\underline{i} + 3xy\,\underline{j} + (x+y)\,\underline{k}.$$

a unit normal vector to S' is $\underline{k}$ (oriented upwards)

so $\iint_{S'} (\text{curl}\,\underline{F})\cdot\underline{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} (-x^2\,\underline{i} + 3xy\,\underline{j} + (x+y)\underline{k})\cdot\underline{k}\,dx\,dy$

$$= \int_{-1}^{1}\int_{-1}^{1} (x+y)\,dx\,dy = \int_{-1}^{1}\left[\tfrac{1}{2}x^2 + xy\right]_{-1}^{1} dy$$
$$= \int_{-1}^{1} \left(\tfrac{1}{2} + y - (\tfrac{1}{2} - y)\right) dy = \int_{-1}^{1} 2y\,dy = y^2\Big|_{-1}^{1} = 1 - 1 = 0.$$

17. check $\oint_C \underline{F} \cdot d\underline{r}$ (alternative method)




$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$C_1$: $\underline{r}(t) = (1-t)\underline{i} + \underline{j} - \underline{k} \quad (0 \le t \le 2)$

$\underline{r}'(t) = -\underline{i}$

$\underline{F}(\underline{r}(t)) = -(1-t)\underline{i} + (1-t)\underline{j} - (1-2t+t^2)\underline{k}$

$\Rightarrow \int_{C_1} \underline{F} \cdot d\underline{r} = \int_0^2 \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$

$= \int_0^2 (1-t)\, dt = \left[t - \frac{1}{2}t^2\right]_0^2$

$= 2 - 2 = 0$.

$C_2$: $\underline{r}(t) = -\underline{i} + (1-t)\underline{j} - \underline{k} \quad 0 \le t \le 2$.

$\underline{r}'(t) = -\underline{j}$

$\underline{F}(\underline{r}(t)) = (1-t)\underline{i} - (1-t)\underline{j} - (1-t)\underline{k}$

$\Rightarrow \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t) = 1-t$

$\Rightarrow \int_{C_2} \underline{F} \cdot d\underline{r} = \int_0^2 (1-t)\, dt = 0$

$C_3$: $\underline{r}(t) = (t-1)\underline{i} - \underline{j} - \underline{k}$, $0 \le t \le 2$

$\underline{r}'(t) = \underline{i}$

$\underline{F}(\underline{r}(t)) = (t-1)\underline{i} - (t-1)\underline{j} + (t-1)^2\underline{k}$.

$\underline{F}(\underline{r}(t)) \cdot \underline{r}'(t) = t-1$

$\int_{C_3} \underline{F} \cdot d\underline{r} = \int_0^2 (t-1)\, dt = \frac{1}{2}t^2 - t\Big|_0^2 = 0$.

$C_4$: $\underline{r}(t) = \underline{i} + (t-1)\underline{j} - \underline{k} \quad , 0 \le t \le 2$.

$\underline{r}'(t) = \underline{j}$

$\underline{F}(\underline{r}(t)) = -(t-1)\underline{i} + (t-1)\underline{j} - (t-1)\underline{k}$.

$\underline{F}(\underline{r}(t)) \cdot \underline{r}'(t) = t-1$

$\Rightarrow \int_0^2 (t-1)\, dt = \frac{1}{2}t^2 - t\Big|_0^2 = 0$.

$\Rightarrow \oint_C \underline{F} \cdot d\underline{r} = 0 + 0 + 0 + 0 = 0$.

18. $A\underline{x} = L(L^T\underline{x}) = \underline{b}$

Set $\underline{y} = L^T\underline{x}$ & first solve

$L\underline{y} = \underline{b}$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 10 \end{pmatrix}$$

$\Rightarrow \quad y_1 = 1$
$y_2 = -1$
$y_3 = 0$

Now solve $L^T\underline{x} = \underline{y}$ for $\underline{x}$.

$$\begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\Rightarrow x_3 = 0$

$\Rightarrow x_2 = -1/4$

$2x_1 - \frac{1}{4} = 1 \quad \Rightarrow x_1 = \frac{5}{8}.$

$$\Rightarrow \underline{x} = \begin{pmatrix} 5/8 \\ -1/4 \\ 0 \end{pmatrix}$$

also,

$\Rightarrow \det A = \det(LL^T)$
$= \det(L)\det(L^T)$
$= (2\times4\times5)\times(2\times4\times5)$
$= 40 \times 40 = 1600.$

19. $A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 4 & 0 & 2 & 1 \\ 8 & 9 & 6 & 3 \\ 4 & 6 & 4 & 2 \end{pmatrix}$ use row operations...

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - R_1$

$$\Rightarrow \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & -3 & 0 & 0 \\ 2 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 \end{pmatrix} \quad \begin{array}{l} R_3 \to R_3 - (-1)R_2 \\ R_4 \to R_4 - (-1)R_2 \end{array}$$

$$\to \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & -3 & 0 & 0 \\ 2 & -1 & 2 & 1 \\ 1 & -1 & 2 & 1 \end{pmatrix} \quad R_4 \to R_4 - R_3$$

$$\to \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & -3 & 0 & 0 \\ 2 & -1 & 2 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = LU$$

Here we have used the notation: $n$ in the i-j entry means we used the elementary row operation

$$R_i \to R_i - nR_j$$

to make the i-j entry zero. An entry with $n$ is really zero, but we use this notation to conveniently read off the off-diagonal entries of $L$. For how this works, see lectures.

Also, $\det A = \det(L)\det(U) = 1 \times (4 \times -3 \times 2 \times 0) = 0$.

20. Use similar notation to previous question.

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 3 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad R_3 \leftrightarrow R_4$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

We have $LU = P_{34} P_{23} A$

where $P_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $P_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$\Rightarrow A = P_{23} P_{34} L \, U \Rightarrow \det A = \det(P_{23}P_{34}) \det(L) \det(U)$

$= (-1)^2 \times 1 \times (1 \times 1 \times 1 \times 1) = 1$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{P = P_{23}P_{34}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{U}$$

21. $\cos y \sinh x + 1 - \sin y \cosh x \dfrac{dy}{dx} = 0.$

Set $P(x,y) = \cos y \sinh x + 1$

& $Q(x,y) = -\sin y \cosh x$

$$\left.\begin{aligned} \Rightarrow \frac{\partial P}{\partial y} &= -\sin y \sinh x \\ \& \ \frac{\partial Q}{\partial x} &= -\sin y \sinh x \end{aligned}\right\} \Rightarrow \text{ODE is exact.}$$

$\Rightarrow \exists\, f(x,y)$ s.t. $\dfrac{\partial f}{\partial x} = \cos y \sinh x + 1$

$\Rightarrow f(x,y) = \cos y \cosh x + x + g(y)$

$\Rightarrow \dfrac{\partial f}{\partial y} = -\sin y \cosh x + g'(y) = -\sin y \cosh x$

$\Rightarrow g(y) = \text{const.}$

$\Rightarrow$ implicit solution is $\cos y \cosh x + x = k$

22. $e^{-2\theta}\frac{dr}{d\theta} - 2re^{-2\theta} = 0$

Set $P(r,\theta) = -2re^{-2\theta}$
$Q(r,\theta) = e^{-2\theta}$

$$\left.\begin{aligned}\frac{\partial P}{\partial r} &= -2e^{-2\theta}\\ \frac{\partial Q}{\partial \theta} &= -2e^{-2\theta}\end{aligned}\right\} \Rightarrow \text{O.D.E. is exact}$$

$\Rightarrow \exists f(r,\theta)$ s.t. $\frac{\partial f}{\partial \theta} = -2re^{-2\theta}$

$\Rightarrow f(r,\theta) = re^{-2\theta} + g(r)$

$\Rightarrow \frac{\partial f}{\partial r} = e^{-2\theta} + g'(r)$

$= Q(r,\theta)$

$\Rightarrow g'(r) = 0 \Rightarrow g(r) = \text{const.}$

$\Rightarrow$ implicit solution is

$re^{-2\theta} = k$

$\Rightarrow$ explicit general solution is

$r(\theta) = ke^{2\theta}$.

(Note: multiply O.D.E by $e^{2\theta}$

$\Rightarrow \frac{dr}{d\theta} - 2r = 0$ which is separable!

$\Rightarrow \int \frac{dr}{r} = 2\int d\theta \Rightarrow \ln|r| = 2\theta + c$.

$\Rightarrow r = ke^{2\theta}$.

Initial condition $r(0) = 1 \Rightarrow k = 1$

$\Rightarrow$ solution is $r(\theta) = e^{2\theta}$.

23. $3x^2-2xy+3y^2-2x-2y-4=0$

$$\Rightarrow (x \quad y)\underbrace{\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}}\begin{pmatrix}x\\ y\end{pmatrix}-(2 \quad 2)\begin{pmatrix}x\\ y\end{pmatrix}-4=0$$

$A$ is symmetric $\Rightarrow$ orthogonally diagonalisable.

Eigenvalues of $A$ satisfy $(3-\lambda)(3-\lambda)-1=0$

$$\Rightarrow 9-6\lambda+\lambda^2-1=0$$

$$\Rightarrow \lambda^2-6\lambda+8=0 \Rightarrow (\lambda-4)(\lambda-2)=0$$

$$\Rightarrow \lambda=4,2$$

$\underline{\lambda=4}$: $\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow a=-b \Rightarrow$ eigenvector is $b\begin{pmatrix}-1\\ 1\end{pmatrix}$

normalise $\longrightarrow \begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$

$\underline{\lambda=2}$: $\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow a=b \Rightarrow$ eigenvector is $b\begin{pmatrix}1\\ 1\end{pmatrix}$

normalise $\longrightarrow \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$

Form $P=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$ which must be orthogonal since columns are unit eigenvectors of a symmetric matrix and hence form an orthonormal set.

$\Rightarrow P^{-1}=P^T=\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$ & check $\det(P)=1$
($\Rightarrow$ rotation in $\mathbb{R}^2$)

$\Rightarrow A=PDP^T$, $D=\begin{pmatrix}2 & 0\\ 0 & 4\end{pmatrix}$

$\Rightarrow$ $\underline{x}^TA\underline{x}=\underline{x}^TPDP^T\underline{x}$

Introduce new variables $\begin{pmatrix}u\\ v\end{pmatrix}=\underline{v}=P^T\underline{x}=\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

$$=\begin{pmatrix}(x+y)/\sqrt{2}\\ (-x+y)/\sqrt{2}\end{pmatrix}$$

& $\underline{x}=P\underline{v}=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}(u-v)/\sqrt{2}\\ (u+v)/\sqrt{2}\end{pmatrix}$

Also $(2 \quad 2)\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=(4/\sqrt{2} \quad 0)=(2\sqrt{2} \quad 0)$

$\Rightarrow$ equation becomes $\underline{v}^TD\underline{v}-(2 \quad 2)P\underline{v}-4=0$

$$\Rightarrow 2u^2+4v^2-2\sqrt{2}\,u-4=0$$

$$\Rightarrow 2\left(u^2 - \sqrt{2}\,u + \tfrac{1}{2}\right) + 4v^2 - 4 - 1 = 0$$

(here we completed the square)

$$\Rightarrow 2\left(u - \frac{\sqrt{2}}{2}\right)^2 + 4v^2 = 5$$

Set $s = u - \frac{\sqrt{2}}{2}$, $t = v$

$\Rightarrow$ equation is $\dfrac{s^2}{(5/2)} + \dfrac{t^2}{5/4} = 1$

which describes an ellipse.

24. $y'' + 2y' + y = 2\cosh x \quad , \quad y(0) = \frac{3}{4} \,,\, y'(0) = \frac{1}{4}$.

General solution is of the form $y = y_H + y_p$
where $y_H$ is the general solution of
$y'' + 2y' + y = 0$

$\Rightarrow$ characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$
$\Rightarrow (\lambda+1)^2 = 0$
$\Rightarrow \lambda = -1$.

$\Rightarrow y_H = Ae^{-x} + Bxe^{-x}$.

For $y_p$, use method of undetermined coefficients.

Note $2\cosh x = e^x + e^{-x}$.

$\Rightarrow$ guess $y_p = ae^x + bx^2e^{-x}$ (since $e^{-x}, xe^{-x}$ appear in $y_H$)

$y_p' = ae^x + 2bxe^{-x} - bx^2e^{-x}$
$y_p'' = ae^x + 2be^{-x} - 4bxe^{-x} + bx^2e^{-x}$

$$
\begin{aligned}
y_p'' + 2y_p' + y_p &= ae^x + 2be^{-x} - 4bxe^{-x} + bx^2e^{-x} \\
&\quad + 2ae^x + 4bxe^{-x} - 2bx^2e^{-x} \\
&\quad + ae^x + bx^2e^{-x} \\
&= 4ae^x + 2be^{-x} \\
&= RHS = e^x + e^{-x}
\end{aligned}
$$

equate coefficients $\Rightarrow 4a = 1 \,,\, 2b = 1 \Rightarrow a = \frac{1}{4}, b = \frac{1}{2}$

$\Rightarrow y = Ae^{-x} + Bxe^{-x} + \frac{1}{4}e^x + \frac{1}{2}x^2e^{-x}$

$y(0) = 3/4 = A + 1/4 \Rightarrow A = 1/2$.

$y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} + 1/4e^x + xe^{-x} - \frac{1}{2}x^2e^{-x}$

$y'(0) = 1/4 = -A + B + 1/4$
$\Rightarrow B = A = 1/2$

$\Rightarrow y(x) = \frac{1}{2}e^{-x} + \frac{1}{2}xe^{-x} + \frac{1}{2}x^2e^{-x} + \frac{1}{4}e^x$

25. $6\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + y = t - 10\sin t$, $y(0)=9$, $\frac{dy}{dt}(0) = 4$

General solution is of the form $y = y_H + y_P$, where $y_H$ is the general solution of $6\ddot{y} - 5\dot{y} + y = 0$.

$\Rightarrow$ characteristic equation is $6\lambda^2 - 5\lambda + 1 = 0$

$$\Rightarrow (1-3\lambda)(1-2\lambda) = 0$$

$$\Rightarrow \lambda = 1/3, 1/2$$

$$\Rightarrow y_H = Ae^{t/3} + Be^{t/2}$$

For $y_P$, use method of undetermined coefficients

guess $y_p = a + bt + c\cos t + d\sin t$

$\dot{y}_p = b - c\sin t + d\cos t$

$\ddot{y}_p = -c\cos t - d\sin t$

$$6\ddot{y}_p - 5\dot{y}_p + y_p = -6c\cos t - 6d\sin t - 5b + 5c\sin t - 5d\cos t + a + bt + c\cos t + d\sin t$$

$$= a - 5b + bt - 5(c+d)\cos t + 5(c-d)\sin t$$

$$= \text{RHS} = t - 10\sin t.$$

equate coefficients $\Rightarrow$ $b = 1$

$a - 5b = 0 \Rightarrow a = 5.$

$c + d = 0 \Rightarrow d = -c$

$\Rightarrow 5(c-d) = -10$

$\Rightarrow 10c = -10$

$\Rightarrow c = -1.$

$\Rightarrow d = 1.$

$$\Rightarrow y = Ae^{t/3} + Be^{t/2} + 5 + t - \cos t + \sin t$$

$y(0) = 9 = A + B + 5 - 1 \Rightarrow A + B = 5$

$\dot{y} = \frac{1}{3}Ae^{t/3} + \frac{1}{2}Be^{t/2} + 1 + \sin t + \cos t.$

~~$\dot{y}(0) = 4$~~ $\dot{y}(0) = 4 = \frac{A}{3} + \frac{B}{2} + 2.$

$\Rightarrow 2A + 3B = 12$

& $A + B = 5$

$\Rightarrow B = 2, A = 3.$

$$\Rightarrow y(t) = 3e^{t/3} + 2e^{t/2} + 5 + t - \cos t + \sin t.$$

28.26. $y'' + 4y' + 4y = 6te^{-2t}$, $y(0) = -1$, $y'(0) = 2$

General solution is of the form $y = y_H + y_p$, where $y_H$ is the general solution of $y'' + 4y' + 4y = 0$

$\Rightarrow$ characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$

$$\Rightarrow (\lambda+2)^2 = 0$$

$$\Rightarrow \lambda = -2.$$

$$\Rightarrow y_H = Ae^{-2t} + Bte^{-2t}$$

For $y_p$, not sure what to guess, so just use variation of parameters.

$$y_p = u(t)\, y_1(t) + v(t)\, y_2(t)$$

where $y_1 = e^{-2t}$, $y_2 = te^{-2t}$

$$u = -\int \frac{y_2 r}{W}\, dt, \quad v = \int \frac{y_1 r}{W}\, dt$$

$$r = 6te^{-2t}, \quad W = y_1 y_2' - y_1' y_2$$

$$= e^{-2t}(e^{-2t} - 2te^{-2t}) - (-2e^{-2t})te^{-2t}$$

$$= e^{-4t}$$

$$u = -\int \frac{te^{-2t} \cdot 6te^{-2t}}{e^{-4t}}\, dt = -\int 6t^2\, dt = -2t^3$$

$$v = \int \frac{e^{-2t} \cdot 6te^{-2t}}{e^{-4t}}\, dt = \int 6t\, dt = 3t^2$$

$$\Rightarrow y_p = -2t^3 \cdot e^{-2t} + 3t^2 \cdot te^{-2t} = t^3 e^{-2t}.$$

$$\Rightarrow y = Ae^{-2t} + Bte^{-2t} + t^3 e^{-2t}.$$

$$y(0) = -1 = A$$

$$y' = -2Ae^{-2t} + Be^{-2t} - 2Bte^{-2t} + 3t^2 e^{-2t} - 2t^3 e^{-2t}.$$

$$y'(0) = 2 = -2A + B. \quad \Rightarrow B = 0$$

$$\Rightarrow y(t) = -e^{-2t} + t^3 e^{-2t}.$$

27. $y_1 = x^2$ is a solution to
$x^2y'' + 2xy' - 6y = 0$.

Set $y_2 = ux^2$, where $u \equiv u(x)$, substitute $y_2$ into ODE to determine $u(x)$ such that $y_2$ is a solution:

$y_2' = u'x^2 + 2ux$

$y_2'' = u''x^2 + 4u'x + 2u$.

$$\Rightarrow x^2y_2'' + 2xy_2' - 6y_2 = u''x^4 + 4u'x^3 + 2ux^2 + 2u'x^3 + 4ux^2 - 6ux^2$$

$$= u''x^4 + 6u'x^3$$

$$= 0 \quad \text{(RHS)}.$$

set $v = u' \Rightarrow v'x^4 + 6vx^3 = 0$

$\Rightarrow v'x + 6v = 0$.

(separable) $\Rightarrow \int \frac{1}{v}\frac{dv}{dx}\,dx = -6\int \frac{dx}{x}$

$\Rightarrow \ln|v| = -6\ln|x|$

$= \ln(x^{-6})$

$\Rightarrow v = x^{-6}$

$\Rightarrow u = \int \frac{dx}{x^6} = \frac{-1}{5x^5}$

Sufficient to take $\frac{1}{x^5} = u$.

$\Rightarrow y_2 = \frac{1}{x^5}\cdot x^2 = \frac{1}{x^3}$

$\Rightarrow$ general solution is

$$y = Ax^2 + \frac{B}{x^3}.$$

28 Set $u = x^2$

$\Rightarrow$ in $\int$, $du = 2x\,dx$.

$$\Rightarrow \int \frac{3x\,dx}{\sqrt{x^4-9}} = \int \frac{\frac{3}{2}\,du}{\sqrt{u^2-9}}$$

set $u = 3\cosh t$

$\Rightarrow$ in $\int$, $du = 3\sinh t\,dt$.

$$\& \quad u^2 - 9 = 9\cosh^2 t - 9$$
$$= 9\sinh^2 t$$

So integral becomes

$$\longrightarrow \int \frac{\frac{3}{2}\cdot 3 \sinh t\,dt}{\sqrt{9\sinh^2 t}}$$

$$= \int \frac{9/2}{3}\,dt = \frac{3}{2}t + C$$

$$= \frac{3}{2}\cosh^{-1}\left(\frac{u}{3}\right) + C$$

$$\Rightarrow \int \frac{3x\,dx}{\sqrt{x^4-9}} = \frac{3}{2}\cosh^{-1}\left(\frac{x^2}{3}\right) + C.$$

(assuming $x^4 - 9 > 0$)

29. Set $x = 4\sinh u$

$\Rightarrow$ in $\int$, $dx = 4\cosh u \, du$

$$\Rightarrow \int \frac{4\,dx}{\sqrt{x^2+16}} = \int \frac{16\cosh u \, du}{\sqrt{16\sinh^2 u + 16}}$$

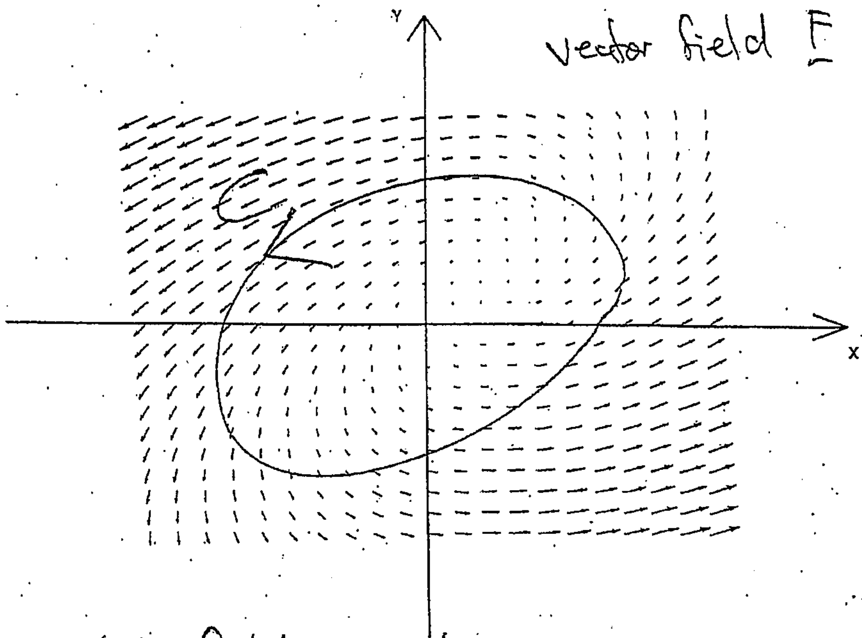
$$= \int \frac{16\cosh u \, du}{4\cosh u} \quad (\text{since } \cosh^2 u - \sinh^2 u = 1)$$

$$= 4u + C$$

$$= 4\sinh^{-1}\left(\frac{x}{4}\right) + C$$

~~32~~.30.

vector field $\underset{\sim}{F}$



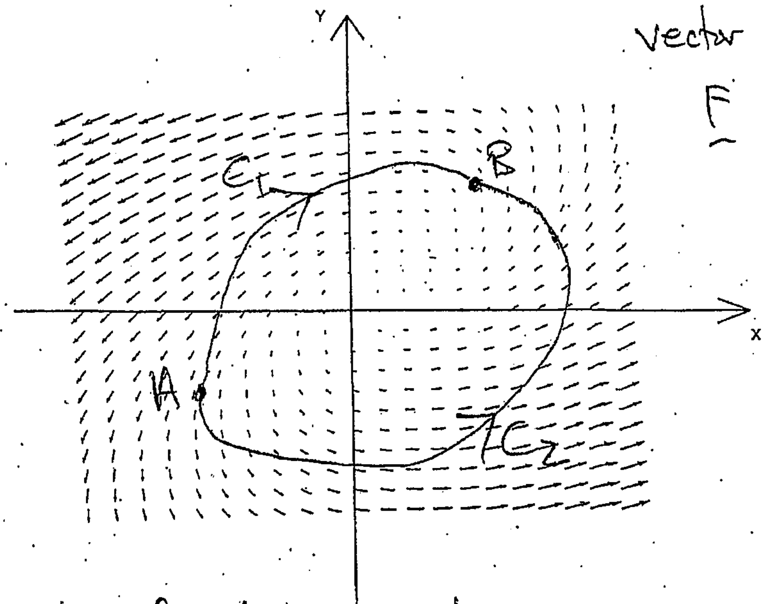

The vector field is <u>not</u> conservative.

A vector field is conservative if and only if line integrals over <u>every</u> closed curve are zero. In the case of the curve C (drawn in the diagram as a counter example), the vector field seems to have components tangent to ~~the curve in the d~~ and in the same direction as C at every point on C. Hence for this case,

$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} \neq 0 \quad (\text{or} > 0)$$

$\Rightarrow \underset{\sim}{F}$ is not conservative.

30. (alternative solution)




vector field $\underset{\sim}{F}$

The vector field is <u>not</u> conservative.

A vector field is conservative if and only if line integrals between two fixed points are path independent. Let $C_1$ and $C_2$ be two distinct curves between the points $A$ and $B$ as in the diagram above.

We will have $\int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} < 0$ since at every point on $C_1$, the vector field has a tangent component in the opposite direction to $C_1$. Also, $\int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} > 0$ since at every point on $C_2$ the vector field has a tangent component in the same direction as $C_2$. Hence $\int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} \neq \int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r}$

$\Rightarrow$ line integrals are path dependent $\Rightarrow \underset{\sim}{F}$ is not conservative.