DEPARTMENT OF MATHEMATICS

MATH2000
Cylindrical and Spherical Coordinates solutions.

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(1) See that

$$\begin{aligned} x^2+y^2+z^2 &= (r\cos\theta\sin\phi)^2+(r\sin\theta\sin\phi)^2+(r\cos\phi)^2 \\ &= r^2\left(\sin^2\phi(\cos^2\theta+\sin^2\theta)+\cos^2\phi\right) \\ &= r^2\left(\sin^2\phi(1)+\cos^2\phi\right)=r^2 \end{aligned}$$

so

$$r=\sqrt{x^2+y^2+z^2}.$$

Also

$$\frac{y}{x}=\frac{r\sin\theta\sin\phi}{r\cos\theta\sin\phi}=\frac{\sin\theta}{\cos\theta}=\tan\theta$$

$$\theta=\tan^{-1}\frac{y}{x}.$$

$$\phi=\cos^{-1}\frac{z}{r}=\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

(2) (a) Note that $x^2+y^2+z^2=r^2 \Rightarrow$ in spherical coordinates the surface becomes $r=3$, which is the equation of a sphere centred at the origin.

(b) Note that $x^2+y^2=r^2\sin^2\phi$ in spherical coordinates.

$$\begin{aligned} \Rightarrow\ z=r\cos\phi &= \sqrt{3}\,r\sin\phi \\ \Rightarrow\ \tan\phi &= \frac{1}{\sqrt{3}} \\ \Rightarrow\ \phi &= \frac{\pi}{6}, \end{aligned}$$

which is the equation of a cone symmetric about the z-axis.

(c) $y=x$ becomes $r\sin\theta\sin\phi=r\cos\theta\sin\phi$ in spherical coordinates.

$$\begin{aligned} \Rightarrow\ \tan\theta &= 1 \\ \Rightarrow\ \theta &= \frac{\pi}{4}, \end{aligned}$$

which is the equation of a plane containing the z-axis.

(d) $z=h$ becomes $r\cos\phi=h$ which is the equation of a plane parallel to the x-y plane.

(e) $x^2+y^2=r^2\sin^2\phi=4\Rightarrow r\sin\phi=2$, which is the equation of a cylinder symmetric about the z-axis.

(3)

$$M = \iiint \rho \; dx \; dy \; dz = \int_0^{\pi} \int_0^{2\pi} \int_a^b \rho(x, y, z) \; r^2 \sin\phi \; dr \; d\theta \; d\phi$$
$$= \int_0^{\pi} \int_0^{2\pi} \int_a^b k(r\cos\phi)^2 r^2 \sin\phi \; dr \; d\theta \; d\phi$$
$$= \left(\int_0^{\pi} \cos^2\phi \sin\phi \; d\phi\right) \left(\int_0^{2\pi} d\theta\right) \left(\int_a^b kr^4 \; dr\right)$$

Take $u = \cos\phi$ in ϕ integral, so $du = -\sin\phi \; d\phi$, $u(0) = \cos 0 = 1$ and $u(\pi) = \cos\pi = -1$ so we have

$$M = \int_1^{-1} u^2(-du) \times \theta|_0^{2\pi} \times \frac{k}{5} r^5|_a^b$$
$$\Rightarrow \; M = \frac{1}{3} u^3|_{-1}^{1} \times 2\pi \times \frac{k}{5}(b^5 - a^5)$$

(Since $-\int_1^{-1} f(u)du = \int_{-1}^{1} f(u)du$)

$$\Rightarrow \; M = \frac{1}{3} \times 2 \times 2\pi \times \frac{k}{5}(b^5 - a^5)$$

$$\Rightarrow \; M = \frac{4\pi k}{15}(b^5 - a^5)$$

(4) For this problem it is best to consider the volume in two parts, using cylindrical coordinates: The cylinder part and the curved part intersected by the parabaloid.
For the cylinder, r ranges from 0 to 1, with θ from 0 to 2π. The maximum z value on the parabaloid is 1, so z can range from 1 to 4.
For the other part, r will only range from the surface of the parabaloid to 1. θ will still go from 0 to 2π and z will be restricted from 0 (where the parabaloid and cylinder intersect) and 1.

$$z = 1 - x^2 - y^2$$
$$r^2 = x^2 + y^2$$
$$\Rightarrow z = 1 - r^2$$
$$\Rightarrow r = \sqrt{1 - z}$$

The density, ρ, is proportional to r, so let $\rho = kr$.

$$\begin{aligned}
\text{mass} &= \int_0^{2\pi}\int_0^1\int_{\sqrt{1-z}}^1 kr^2\ dr\ dz\ d\theta + \int_0^{2\pi}\int_1^4\int_0^1 kr^2\ dr\ dz\ d\theta \\
&= 2\pi\int_0^1 [\frac{1}{3}kr^3]_{\sqrt{1-z}}^1 dz + 2\pi\times 3\times\frac{1}{3}kr^3|_0^1 \\
&= 2\pi\int_0^1 (\frac{1}{3}k - \frac{1}{3}k(1-z)^{\frac{3}{2}})dz + 2\pi k \\
&= 2\pi k([\frac{1}{3}z - \frac{1}{3}\frac{2}{5}(1-z)^{\frac{5}{2}}]_0^1 + 1) \\
&= 2\pi k(\frac{1}{3} - \frac{2}{15} + 1) \\
&= \frac{12\pi k}{5}.
\end{aligned}$$

As an exercise, try changing the order of integration (using the method shown in lectures) of the integral from lectures by swapping dr and dz. Make sure you get the sum of the two integrals obtained above.

(5) Use spherical polar coordinates i.e. $x = r\cos\theta\sin\phi, y = r\sin\theta\sin\phi, z = r\cos\phi$, and $|J| = r^2\sin\phi$. The density function will be represented by $\rho = \rho_0(1+\frac{r}{a})$. Using polar coordinates, the region is

$$V = \{(\phi,\theta,r)|0\le\phi\le\frac{\pi}{2}, 0\le\theta\le 2\pi, 0\le r\le a\}$$

Note that $0\le\phi\le\frac{\pi}{2}$ since we are describing the hemisphere with $z\ge 0$.
Let m denote the mass.

$$\begin{aligned}
m &= \int\int\int_V \rho dV \\
&= \int^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^a \rho_0(1+\frac{r}{a})r^2\sin\phi dr d\theta d\phi \\
&= \rho_0\int_0^{\frac{\pi}{2}}\sin\phi d\phi\int_0^{2\pi}d\theta\int_0^a(r^2+\frac{r^3}{a})dr \\
&= \rho_0[-\cos\phi]_0^{\frac{\pi}{2}}\times[\theta]_0^{2\pi}\times[\frac{r^3}{3}+\frac{r^4}{4a}]_0^a \\
&= \rho_0(0+1)2\pi(\frac{7a^3}{12}) \\
&= \frac{7\pi\rho_0 a^3}{6}
\end{aligned}$$

Using symmetry, we know that $\bar{x}, \bar{y} = 0$, so we only need to find $\bar{z}$ using M_{xy}.

$$\begin{aligned}
M_{xy} &= \int\int\int_V z\rho dV \\
&= \int^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^a r\cos\phi\rho_0(1+\frac{r}{a})r^2\sin\phi dr d\theta d\phi \\
&= \rho_0 \int_0^{\frac{\pi}{2}} \cos\phi\sin\phi d\phi \int_0^{2\pi} d\theta \int_0^a (r^3 + \frac{r^4}{a})dr
\end{aligned}$$

For $\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi d\phi$, let $u = \sin\phi$, $\frac{du}{dx} = \cos\phi$, so $\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi d\phi = \int_{u=0}^{u=1} udu = [\frac{u^2}{2}]_0^1 = \frac{1}{2}$.

$$\begin{aligned}
\Rightarrow M_{xy} &= \rho_0 \times \frac{1}{2} \times [\theta]_0^{2\pi} \times [\frac{r^4}{4} + \frac{r^5}{5a}]_0^a \\
&= \rho_0 \times \frac{1}{2} \times 2\pi \times a^4(\frac{1}{4} + \frac{1}{5}) \\
&= \frac{9}{20}\pi\rho_0 a^4
\end{aligned}$$

Hence $\bar{z} = \frac{M_{xy}}{M} = \frac{27a}{70}$, and the centre of mass is located at $(0, 0, \frac{27a}{70})$.

(6) (a) Cylindrical coordinates are $x = r\cos\theta$, $y = r\sin\theta$ with the usual z coordinate. The two paraboloids bounding the region are $z = r^2$ and $z = 36 - 3r^2$, which intersect in the plane $z = 9$ at $r = 3$ (ie. a circle of radius 3 centred about the z-axis and lying in the plane $z = 9$).

The region of integration can be expressed as

$$0 \le r \le 3,\ 0 \le \theta \le 2\pi,\ r^2 \le z \le 36 - 3r^2.$$

To calculate the volume, we have

$$\begin{aligned}
\text{vol.} &= \iiint_E dV \\
&= \int_0^3 \int_0^{2\pi} \int_{r^2}^{36-3r^2} r\ dz\ d\theta\ dr \\
&= 2\pi \int_0^3 r(36 - 3r^2 - r^2)dr \\
&= 2\pi \int_0^3 (36r - 4r^3)dr \\
&= 2\pi \left[18r^2 - r^4\right]_0^3 \\
&= 162\pi.
\end{aligned}$$

(b) For the centroid (assume the density is constant $= k$), by symmetry $\overline{x} = \overline{y} = 0$ (you could check this by direct calculation), with z coordinate

$$\begin{aligned}
\overline{z} &= \frac{\iiint_E kz \; dV}{\iiint_E k \; dV} \\
&= \frac{1}{162\pi} \int_0^3 \int_0^{2\pi} \int_{r^2}^{36-3r^2} rz \; dz \; d\theta \; dr \\
&= \frac{1}{162\pi} \times 2\pi \times \int_0^3 r \left[\frac{1}{2} z^2\right]_{r^2}^{36-3r^2} \\
&= \frac{1}{162} \int_0^3 \left(1296r - 216r^3 + 8r^5\right) dr \\
&= \frac{1}{162} \left[648r^2 - 54r^4 + \frac{8}{6} r^6\right]_0^3 \\
&= \frac{1}{162}(5832 - 4374 + 972) = 15.
\end{aligned}$$